

THE MATHEMATICAL GAZETTE

EDITED BY
W. J. GREENSTREET, M.A.

WITH THE CO-OPERATION OF
F. S. MACAULAY, M.A., D.Sc., F.R.S.
AND
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PARTIAL FAILURE OF EUCLID (I. 4) IN TIME-SPACE THEORY.

BY ALFRED A. ROBB, Sc.D., F.R.S.

In the *Mathematical Gazette* of October 1928, page 234, there appears a quotation from a letter of Chief Baron Pollock to his son, when an undergraduate at Trinity, in which, speaking of Euclid, he says : " I consider (I. 4) to be much more of a ' pons asinorum ' than the 5th. Indeed I regard it as the test of a geometric spirit. If it be fairly conquered, it proves the existence of that logical accuracy which is the soul of mathematics, and to elicit and cultivate which is the great benefit which such studies confer as a branch of education."

I am afraid that nowadays competent mathematicians would no longer agree with the Chief Baron.

The method of superposition employed by Euclid will not stand close criticism ; although it is, I fear, too often regarded as good enough for school-boys. This makes it all the more important to call attention to a definite instance in which the theorem breaks down in Time-Space Theory, which, so far as I am aware, is not generally known.

It is true that I referred to this point in my *Theory of Time and Space* (p. 344), published in 1914, in which the subject was treated in a purely geometrical manner, and I also mentioned it in my *Absolute Relations of Time and Space* (p. 69), but I purpose here to consider it from the analytical standpoint.

It may be premised that in the simple type of Time-Space Theory the axis of t behaves as a pure imaginary, and, as a result of this, the condition of normality of the lines

$$\frac{x-a_1}{l} = \frac{y-b_1}{m} = \frac{z-c_1}{n} = \frac{t-d_1}{p};$$

and

$$\frac{x-a_2}{l'} = \frac{y-b_2}{m'} = \frac{z-c_2}{n'} = \frac{t-d_2}{-p'};$$

becomes

$$ll' + mm' + nn' - pp' = 0.$$

The expression

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 - (t_1 - t_2)^2$$

may be either positive, zero, or negative. If the expression be positive and equal to r^2 , then the space-time points (x_1, y_1, z_1, t_1) and (x_2, y_2, z_2, t_2) lie in what I have called a " separation line," and the length of the interval between them is r .

If the expression be negative and equal to $-\bar{r}^2$, then the space-time points lie in what I have called an "inertia line," and the length of the interval is \bar{r} .

From the analytical standpoint an inertia line behaves like a separation line with a pure imaginary length, and *vice versa*, so that \bar{r} is really the modulus; but this is equivalent to doing what we have already done with the axis of t . If the above expression be zero, then the space-time points lie in what I have called an "optical line."

It might be supposed by those who approach the subject from the purely analytical standpoint that the length of the interval in this case should be taken as zero; but this gives a wrong idea of what actually occurs. This is made clear in my *Theory of Time and Space*, and it is there shown that the proper interpretation of the equation

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 - (t_1 - t_2)^2 = 0$$

is merely that the space-time points lie in an optical line as there defined.

Lengths may be compared along the same or along parallel optical lines in just the same way as they may be along the same or along parallel separation or inertia lines; but the comparison of lengths along two optical lines which are oblique to one another becomes meaningless.*

In spite of this seemingly strange behaviour of optical lines, they are much more easily defined geometrically than either separation or inertia lines.

It will be sufficient for our present purpose to consider the plane of x and t , so that the analytical geometry may be taken as two dimensional; the y and z coordinates being taken as zero. Such a plane will contain two optical lines passing through any point (x', t') of it; the equations of these lines being

$$x - x' = \pm(t - t').$$

Suppose now that O be the origin of coordinates and P be any point in the plane whose coordinates x_0, t_0 are such that

$$t_0 = x_0 > 0.†$$

Then obviously this point will lie in the optical line

$$x = t.$$

Now take a point Q whose coordinates x_1, t_1 are such that

$$x_1 = \frac{x_0^2 - g^2}{2x_0},$$

$$t_1 = \frac{x_0^2 + g^2}{2x_0},$$

where g is a real constant other than zero. Then $x_1^2 - t_1^2$ is negative, and so OQ will be an inertia line and

$$OQ^2 = \left(\frac{x_0^2 + g^2}{2x_0} \right)^2 - \left(\frac{x_0^2 - g^2}{2x_0} \right)^2 = g^2.$$

Thus

$$OQ = g \text{ (an inertia line).}$$

The line QP , on the other hand, will be such that $(x_0 - x_1)^2 - (t_0 - t_1)^2$ will be positive and will therefore be a separation line, and we shall have

$$QP^2 = \left(x_0 - \frac{x_0^2 - g^2}{2x_0} \right)^2 - \left(x_0 - \frac{x_0^2 + g^2}{2x_0} \right)^2 = g^2.$$

Thus we shall have

$$QP = g \text{ (a separation line).}$$

* An interesting analogy to this sort of thing in an entirely different domain is the fact, for instance, that the intensities of a pain may be compared if suffered by the same individual, but cannot be compared if suffered by different individuals, although it be descriptively of the same type in both cases.

† The restriction of x_0 to positive values is not really essential, but is convenient for the special purpose in view.

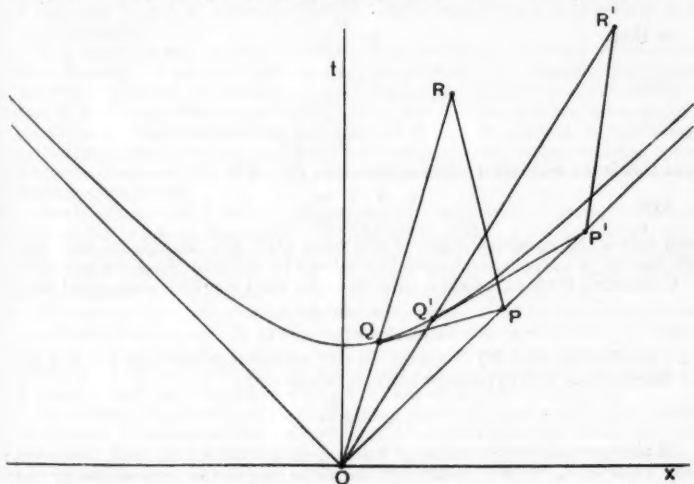
The equation of the line OQ is

$$\frac{x}{\frac{x_0^2 - g^2}{2x_0}} = \frac{t}{\frac{x_0^2 + g^2}{2x_0}}.$$

The equation of the line QP is

$$\frac{x - \frac{x_0^2 - g^2}{2x_0}}{\frac{x_0^2 - g^2}{2x_0}} = \frac{t - \frac{x_0^2 + g^2}{2x_0}}{\frac{x_0^2 + g^2}{2x_0}}.$$

Thus, multiplying corresponding denominators and subtracting, we get zero, so that OQ and QP are normal to one another. But the position of P



may be anywhere on the part of the optical line $x=t$ for which the coordinates have positive values, so that x_0 may have any positive value whatever.

Thus, if we consider the triangle OQP , we see that the side OP for one value of x_0 will be a part of the corresponding side OP' for any greater value of x_0 : so that the triangle is not determined in all its dimensions when the magnitudes of the sides OQ and QP are given, along with the fact that they are normal to one another.

The length OP will be proportional to x_0 , so that the analogue of Euclid (I. 4) does not hold in this case. It is to be noted, however, that this failure only occurs when OP is an optical line. If OP be either an inertia line or a separation line, while the sides OQ and QP are normal to one another and of given lengths, then the length of OP is determinate.

In order to show that there is a similar failure of Euclid (I. 4) when the two given sides of the triangles are not normal to one another, let us next take a point R on OQ whose coordinates x_2, t_2 are given by

$$x_2 = \frac{x_0^2 - g^2}{2x_0} \frac{b}{g},$$

$$t_2 = \frac{x_0^2 + g^2}{2x_0} \frac{b}{g}.$$

Then clearly
or

$$OR^2 = b^2 \\ OR = b.$$

If PR is an inertia line, which it will be if $(x_2 - x_0)^2 - (t_2 - t_0)^2$ is negative, then

$$PR^2 = \left\{ \frac{x_0^2 + g^2 b}{2x_0 g} - x_0 \right\}^2 - \left\{ \frac{x_0^2 - g^2 b}{2x_0 g} - x_0 \right\}^2 \\ = b(b - 2g).$$

Thus $PR = \sqrt{b(b - 2g)}$,
so that PR will be an inertia line provided that $b(b - 2g)$ be positive, and this is independent of x_0 .

Let us suppose then that

$$b(b - 2g) = c^2, \\ g = \frac{b^2 - c^2}{2b}.$$

so that

Then

$$\frac{b - g}{c} = \frac{b^2 + c^2}{2bc} \\ = \frac{1}{2} \left(\frac{b}{c} + \frac{c}{b} \right),$$

and this is the numerical value of the ratio $RQ : RP$.

Also

$$\frac{g}{c} = \frac{1}{2} \left(\frac{b}{c} - \frac{c}{b} \right),$$

and this is the numerical value of the ratio $QP : RP$, though, in this case, the line QP is a separation line and therefore of a different character from RP .

Comparing these expressions with those for the hyperbolic cosine and sine,

$$\cosh A = \frac{1}{2} (e^A + e^{-A}), \\ \sinh A = \frac{1}{2} (e^A - e^{-A}),$$

and recollecting that RQ is normal to QP , we see that the lines RQ and RP behave as lines at a hyperbolic angle A , where

$$A = \log \frac{b}{c}.$$

If then we select any values of b and c , such that $b > c$, and also select any value of x_0 , we may substitute the value of g in the expressions for the coordinates of R and obtain

$$x_2 = b^2 \frac{x_0}{b^2 - c^2} - \frac{1}{4} \frac{b^2 - c^2}{x_0}, \\ t_2 = b^2 \frac{x_0}{b^2 - c^2} + \frac{1}{4} \frac{b^2 - c^2}{x_0}.$$

Thus, for different values of x_0 , we get an infinite set of triangles having a pair of sides equal to b and c , while the included hyperbolic angle $A = \log \frac{b}{c}$, but the third sides will bear to one another the ratios of the corresponding values of x_0 .

Thus Euclid (I. 4) again breaks down.

As in the case previously considered, the length of the third side is quite determinate, provided that it be not an optical line.

The fourth is only one out of a large number of Euclid's propositions which require modification in Time-Space Theory, and, in fact, the construction of the very first proposition is only possible in one out of the three types of plane which occur in this geometry.

A. A. ROBB.

VARIETY OF METHOD IN THE TEACHING OF ARITHMETIC.

By W. F. SHEPPARD, Sc.D.

(Presidential Address to the Mathematical Association, 1929.)

1. THE subject of my address is "Variety of Method in the Teaching of Arithmetic." The opposite of variety is uniformity: and the important question is whether we are to have uniformity or variety. There are some people who urge a procrustean uniformity: that for every process there shall be one prescribed method, *quod semper* (or nearly so), *quod ubique*, et *quod ab omnibus*. This is, to my mind, a pernicious doctrine; but the present is not a proper occasion for examining the arguments by which it is supported. I wish only to put in a plea for variety, with a brief statement of reasons and some examples.

2. I ought to explain that I use the word *method* in a rather wider sense than is usual. I am not referring only to methods of performing particular processes, whether of calculation or of reasoning. I have in mind also our way of thinking of the concrete relations to which these processes correspond: our onlook, which depends on our point of view. The onlook is sometimes rather important; teacher and pupil may nominally be dealing with the same problem, but actually, if they are looking at it in different aspects, some confusion may arise.

On the other hand, I am not going to deal with any broad varieties of method of teaching, such as the method of individual work. I am concerned only with the everyday processes that occur in doing what the children call "sums."

3. Variety of method may be of two kinds, which, according to a common nomenclature, I may call variety vertically and variety horizontally.

By variety vertically I mean that a child is not to be limited throughout the whole of his school life to one particular method of working any specified kind of sum: that, for instance, children of six or eight are not to be compelled, as a matter of course, to use exactly the same method as pupils of sixteen or eighteen. In other words, the method is to be progressive.

By variety horizontally I mean that any particular group of pupils—class or division or whatever you call it—should not be restricted to one method of working any one kind of sum, but should know something of other methods, and should have some freedom of choice of method.

It is in this latter kind of variety that I am specially interested. Variety vertically is largely a question of differences between methods which may really be successive stages in the development of a final method. Variety horizontally is a question of differences between children, which are vastly more important: differences of capacity, differences of temperament, differences of mentality.

Having explained my terms, I have to make some general remarks, dealing only with points that seem relevant to this particular question of variety. I will make these remarks as brief as possible, as I want to get on to illustrative examples.

4. My primary position in the matter is a humble one. I am not out to produce mathematicians, or even arithmeticians. My position is that, for some reason or other, the children have to do a good deal of arithmetic; and I want them to enjoy their arithmetic as much as possible. Some children, for example, will find a natural pleasure in trying different methods of calculation and seeing that they arrive at the same result. Others may find calculation dull: the proper remedy for this is not to try to drill the child into mechanical use of a method that is distasteful to him, but to look about for some other method that is, at any rate, less distasteful. There is plenty of fun to be got out of calculation, if a suitable method is found.

5. But, while considering how to make the arithmetic lesson more pleasant, we must at the same time have an eye to our ultimate aims.

Arithmetic has, or may have, a utility value and a culture value. I am not concerned with utility value: this is a large subject in itself. But there are two items of culture value claimed for arithmetic to which I would specially call attention.

The first is that arithmetic can be used to develop the child's self-confidence. It is one of the subjects in which, when he has a new kind of problem before him, he can put all his strength into the solution of it. He has the stimulus of the fight, and possibly the pleasure of victory. His methods may not be the best, but they are his own. This value of the subject is lost if you lead him at every step: if, for instance, you say: "Our next subject is so-and-so: and I will tell you what you have to do, and show you the quickest and easiest way of doing it."

The other item of culture value of the subject is that it enables the pupil, not only to deal with arithmetical problems of particular types, but also to obtain a better knowledge and understanding of the world in which he lives.

6. How is this culture value to be obtained?

One essential is that by the time the pupil reaches the end of his school study of arithmetic he should have some idea as to what it is all about. I am afraid that this is not always the case. He is able to perform certain numerical processes with tolerable speed and accuracy, and to solve certain kinds of problems by assigning them to the proper chapters of his text-book and applying the proper formulae. But his knowledge is not coordinated.

7. It is the old story of not being able to see the wood for the trees. The pupil's knowledge is limited to individual trees: sometimes merely the particular trees that were blazed by the pioneers in the subject. What he needs is to get outside the wood and see it as a whole.

8. But the simile of the wood is not a perfect one. Arithmetic is not a self-contained subject. It is part of an organic whole: you can isolate it only by severing the organic connections.

Outside the school subject, but vitally connected with it, lies the whole world of quantitative science. The child cannot be expected to understand this: but the older pupil ought to have some idea of it. Even in his school work there are Pisgah-crags which involve some stiff climbing but will give him glimpses of the world outside.

9. It is desirable that this fuller knowledge of the subject should include some understanding of the reasons for the methods used. But this principle must not be stressed too much, especially in the early stages. I have said that one item of culture value, placed to the credit of arithmetic, is that it gives the child an opportunity of attacking a new problem (whether a problem in the technical sense or merely a numerical process) in his own way. If he does this, he is laying the foundation for a more thorough knowledge. But I doubt if the majority of children, at any rate of young children, are prepared to make use of the opportunity. My impression is that the ordinary child wants to know what he (or perhaps, even more, she) has to do, without bothering about the reason. Having learnt what he has to do, and attained tolerable skill in doing it, he wants to get on with the next thing. The children who want to know the reason for the process are few. And, of these, some will be content with being told the reason. The exceptional ones are those who like to puzzle out the whole thing—the process, and the reason for it—by themselves. They are the children who prefer to leave the beaten track and go their own way. If they are not allowed to go on voyages of exploration, they will run up and down the hillocks at the side of the track. They cover more ground, and possibly don't reach their destination so quickly: but they enjoy the getting there, and they may possibly get a better knowledge

of the lie of the land. What I am asking for is, that this attitude should not be repressed.

10. A final point, bearing on what I said with regard to my use of the word "method," is that there should, as far as possible, be harmony, not only between the concrete relations or operations with which we deal and the symbols we use to express them, but also between both of these and the way in which we think of these relations or operations. More briefly, there should be harmony between facts, onlook, and notation.

One implication of this is that we should not be in too great a hurry to replace relations between quantities by relations between numbers. The children will do this for us: they like playing about with number-symbols, without having a clear idea of what their processes mean, and without realising that number by itself is unthinkable. It is for us to see that they do not waste their powers on unrealities.

11. *Variety horizontally.* I have so far been making general observations bearing on the question of variety of method. I come now to the suggestion of variety horizontally: the suggestion that the pupils working in any particular group should not be restricted to one method of dealing with any specified kind of "sum," whether it be a purely numerical process or an arithmetical problem. And I do not expect that the suggestion will obtain your immediate approval.

There are two objections you may make at once. You may say: "We haven't time. There is a great deal of work to be got through, and we can't spare time for frills and fads. In particular, we must give a certain amount of time to class work in calculation. It is a great help, for this, to have standard methods of working." And you may also say that rapid changes from one method to another and then back again will only confuse the child.

My answer to these objections is that I am not an extremist, and that we have to balance one advantage against another. It is quite possible that, for children in bulk, it is a good thing to have a standard method in any particular class. This is a matter for experiment, and I should be interested to know whether experiments on the relative merits of uniformity and variety have actually been made. There have been some valuable experiments on the relative efficiency of different methods of numerical calculation: some of you, for instance, will remember the experiments on methods of subtraction to which Miss Punnett called attention in the *Gazette* a few years ago. But we want a great deal more—and wider—experimenting, and we also need to have some principles laid down by which we are to proceed from the results of experiment to action. What I am urging is more experiment.

Even if we could have an agreed standard system suitable to each stage attained, I still think that some time might usefully be spared from drill in order to enable the children to get a better idea of the result they are aiming at. And this can to some extent be done by letting them see that there are different methods of arriving at the result.

12. *Types of children.* There is another point. I have referred to "children in bulk"; in other words, I have been treating the children of any group as forming what in statistics is called a homogeneous population. But there seems to be some ground for supposing that the differences of arithmetical ability in children are not entirely differences of degree, but are to some extent differences of kind. And the first object I had in view when I fixed on the title of my address was to find out what progress had been made in recent years in defining mental types in this respect, in seeing how to distinguish the type of any individual child, and in ascertaining what methods of teaching or study were best suited to the different types.

On this question I drew blank. And when I asked an eminent psychologist if he could refer me to information on the subject, he said that no good work

had been done on it so far as he knew, and that, on the contrary, almost all the good work about types had been rather in the direction of discrediting their existence.

One cannot question the views of psychologists on matters which lie within their province. But, to take a particular class of cases, it certainly does seem to me that the difference between the way in which a series-visualiser—I will explain presently what I mean by this—thinks about numerical operations and the way in which an ordinary person thinks about them is so fundamental that we are justified in regarding the two persons, for our own purposes, as belonging to different categories, even if we do not call them different types.

If there are these fundamental differences between children, their existence strengthens the argument for having some variety of method in order to give each child an opportunity of choosing the particular method that is most suitable to him. Here is a region for experiment.

I referred to the difference between children as regards their attitude towards explanation of processes. That was a difference of temperament. What I am referring to now is a difference of mentality.

13. *Excursions.* I have compared the child's study of arithmetic to progress through a wood. I propose to conclude with some illustrative excursions or rambles, taking us away from the beaten track. The methods I shall use are not entirely novel: indeed, both here and in my general remarks, I am to some extent saying things that I said, or hinted at, twenty years ago.* But I hope that some of you may find in them some material for experiment.

There are three excursions that suggest themselves to me.

The first ramble begins with asking what we really mean by the symbol for a number, say of three figures. This leads us to consideration of the series-idea of number; of the visualisation of the number-series; and of standardisation of this visualisation. From this we pass to the series of multiples of a unit, and thence to ways of thinking about proportion. Two small points of interest are the arrangement of the constituents in a multiplication or division sum, and the arrangement of division by factors.

The second ramble starts with the principle that our notation should be such that the most important things should have most stress laid on them. This gives a principle for fixing the meaning of the sign of multiplication; and here we link up with what we have previously been considering in reference to the arrangement of a multiplication or division sum. We are also led to see the importance of working calculations from left to right whenever possible. A further application of the principle leads to special methods for dealing with multiplication and division of integers or decimals, but this I have to omit.

The third excursion is a piece of crag-climbing. It deals with rates, i.e. operators such as $\frac{60 \text{ miles}}{1 \text{ hour}}$ or $\frac{28 \text{ apples}}{7 \text{ boys}}$.

14. *First Excursion. Meaning of number-symbol.* The starting-point of our first ramble is this. What, exactly, do you mean by the written symbol

284

or the spoken symbol

"two hundred and eighty-four"?

Most of you, I expect, will say that the symbol means the number which is made up of 2 hundreds 8 tens and 4 ones.

Or, I am afraid, you will say "4 units." If this is your practice, I fervently hope that it will not be long before you alter it. The word "unit" has a definite meaning in physical science and in considering the principles of arithmetic (e.g. "unitary method"), and it is a pity to use it in a quite different sense in elementary arithmetic.

* "Arithmetic," in *Encyclopædia Britannica*, 11th ed., II, 523-542.

Let us then say that 284 means the number made up of 2 hundreds 8 tens and 4 ones. Similarly 84 means the number made up of 8 tens and 4 ones; and 4 means—the number made up of 4 ones. But—what *does* 4, itself, mean?

You must define 4 in some way; and the definition must be on a system different from that on which you define 284. Thus your system of naming numbers is a composite one: you have (ignoring the anomaly of 11 and 12) one system for the spoken numbers one to ten or the written numbers 0 to 9, and another system for the numbers above these.

Cannot we devise a system of interpretation that shall be the same throughout?

We cannot extend the large-number system to the small numbers: can we extend the small-number system to the large numbers?

To answer this, we must see what the system really is for the small numbers. What, ultimately, do we mean by the symbol 4?

It is easy to see that the answer is that 4 is the number which is one more than 3, or the number which comes after 3. Similarly 5 is the number which comes after 4, and so on.

This system can be applied to all numbers, so that, as 4 is the number which comes after 3, so 284 is the number which comes after 283. This is a logically consistent system. If we think of numbers in this way, we have the series-idea of number.

You may say: "This is all very well logically. But in the beginnings of arithmetic we are dealing with young children, and it is absurd to expect them to think of numbers logically."

15. *The series-idea.* The objection is quite sound. But, as it happens, there are a good many people, adults as well as children, who do actually think of numbers in this way, i.e. think of them, or at any rate think of the first few numbers, as forming a series. These are the series-visualisers.

Some of you may not know what I mean by this, so I must explain. Francis Galton discovered, a good many years ago, that some people have a tendency, as regards things which are in their nature serial, to think of them as visualised in a definite arrangement or "form" in space. The days of the week, for instance, are cyclical: so they are visualised on a circle or oval. In my own case, Sunday is on the extreme right: Monday is below it to the left, and the other days follow clockwise till I get back to Sunday. And the visual space allotted to each day is divided up so that I locate my engagements in the proper places on it: or at any rate I used to do this, but my visual memory is becoming unreliable. Many things are visualised in this way: days of the week, days of the month, months of the year, numbers, centuries (for fixing dates), years of one's own life, even letters of the alphabet.

To a good many of you all this may seem perfectly idiotic. But the fact remains that some people do think in this way. The curious thing is that, although this has probably been happening for thousands of years, it was not discovered until about fifty years ago. The reason, apparently, was that visualisers either assumed as a matter of course that everybody else thought in the same way as they did, or else thought it was a peculiarity of their own and were shy of mentioning it.

The important case for our purposes is that of the number-visualisers. With these, the numbers, or at any rate the first few, lie along a line of some sort. This fact gives a great help in thinking of numerical relations and processes. I think even more use might be made of it for teaching purposes. And the methods which are useful for the visualisers themselves might possibly be of some use to non-visualisers also.

It is true that the visualisers are relatively few. But, absolutely, they are not very few. A curious thing is the constancy of their proportion to the

total population. One writer,* just over thirty years ago, gave some interesting figures (mainly U.S.A.) from which I extract the following :

<i>No. of persons examined.</i>	<i>Proportion having number-forms.</i>
332 normal school students	6%
343 miscellaneous adults	7%
360 adults	7%
974 school children aged 10-16	8%
2009	7%

Allowing for the fact that the forms tend to be forgotten if they are not used, 7% is not an extravagant estimate as regards children. There being about seven million school children in England and Wales alone, we may take it that about half a million have number-forms. It is surely worth while considering whether our teaching of number can be better adapted to the mentality of these half-million.

Moreover, these half-million are not all that we have to take into account. There are probably another half-million who, without having a number-form, have a visual form of some sort—days of the month, days of the week, Lord's prayer, alphabet, and so on. And there are a still larger number who, though they have no definite forms, associate increase of number with movement, usually upwards. To all of these the series-idea of number may be expected to make some appeal.

16. *Standardisation.* Number-forms are varied. If, therefore, we are to appeal to them in our teaching, the first thing is to see whether we can get a standard form.

This, I think, is not very difficult. I will take my own number-form as an example. It goes vertically up from 1 to 12, turns to the left and goes slightly downward till it gets to 20; then turns and goes straight up again, turns as before at 112 and 120, and then continues faintly up to 132 and 144. This is of a fairly common general type, but with peculiarities which are rather interesting but on which I must not spend time.

For standardising, we must also bring into account the persons who have not a definite number-form and yet with an increase of number associate some movement in space. Including these, it would appear that for the great majority of visualisers the increase of number is upwards, and that, of the remainder, a considerable number think of the increase as taking place to the right.

Provisionally, therefore, I suggest that the standard direction for numbers should be taken to be upwards in a straight line; the secondary direction being to the right. These, you will at once remark, are the standard directions for graphs; the graphed quantity, which is the really important thing, being measured upwards, and the variate of which it is a function being measured to the right. But you must observe that we are now dealing with the natural numbers, not with continuous magnitude, and also that, if we go up to a large number, the scale must change as we go along the series.

I don't know whether a concrete representation is necessary or even helpful, at any rate for the visualisers; for them, it might be sufficient to regard the series of numbers as existing in space. If you want a concrete representation, you can put pins on a blackboard, or you can lay a row of coins or counters along the edge of a table, and mark the tens and also the fives (since ten is rather a large number for small children to deal with) by pencils or matches; and then think of the whole thing as tilted up vertically—or, for the secondary position, tilted up sideways so as to come horizontally.

You may say that my suggestion of standardisation is inconsistent with my suggestion of individual freedom. But compulsion is one thing, help is

* D. E. Phillips, *American Journal of Psychology*, 8 (1896-7), p. 509.

another. We teach children what to do with figures, and we try to teach them how to reason arithmetically; but we do not give them guidance as to how to think about the things they are talking about—how to construct their mental imagery. The existence of number-forms is evidence of a tendency to think of numbers in a particular way; their irregularity is evidence of the need of guidance. And this guidance should begin as early as possible. Moreover, for class-work, a common language is desirable. But we need more experiment before we can come to a final decision as to details.

17. *Advantages of the method.* This way of thinking of number has some advantages, of which I may mention the following.

(1) There is no possibility of confusion of ordinal and cardinal numbers. The names *one, two, three* . . . given to the coins as we count them are ordinal numbers; but the cardinal numbers are associated either with the groups of coins or with the intervals between coins. Thus the match that marks 5 is not placed opposite the fifth coin, but between the fifth and the sixth: it marks the completion of counting the five coins.

(2) We think of the numbers, at any rate from 1 to 20, by their spoken names, not by their written symbols; and we are thus not tied down prematurely to the denary scale. Some people would regard this as a disadvantage; they would like children to think of seventeen as one-ty-seven. I think it is better that in the early stages every number should have its own separate name and its own individuality. Most young children do not realise that seventeen means seven and ten; and I do not see any necessity for telling them. I should like to see the experiment tried of having as much oral and as little written work as possible in the early stages. I think it would give a more intimate knowledge of number; that it would, for instance, produce a sounder addition-table.

(3) Mental addition of two 2-figure numbers is very simple. Knowing that 8 and 6 make 14, which means that the addition of 6 shifts us from 8 to a certain number in the next ten-group, we see at once that the addition of 6 to 48 shifts us to the corresponding number in the next ten-group, namely, to 54. Similarly to add 48 and 26 we get the successive shifts to 58, 68, and 74.

I find that children who have a difficulty in addition catch on to this method very quickly, even if the successive steps are merely indicated in the air.

(4) The number form is specially useful for helping the child to construct a multiple-table, i.e. a table of the multiples of a number, such as 7, 14, 21 . . . as the multiples of 7. I am surprised to find that even some progressive writers regard the multiplication table as of primary importance. It is useful, of course, for the actual process of multiplication, but I suggest that it ought to be built up from, or after, the multiple-table.

(5) Similarly it is very easy to think of a large multiple of a small number, such as 5 or 7. Thus, when we have got a general mental picture of 28 units, each unit being represented, say, by a coin or a counter, we can think of 28 times 5 as obtained by converting each coin or counter into a hand with 5 fingers. This gives us 14 pairs of hands, or 140 fingers. (This, of course, belongs to a relatively advanced stage in which we are using written symbols.)

On the other hand, the method is not specially appropriate to thinking of a small multiple of a large number. Here we must adopt the ordinary process of breaking up and re-grouping.

(6) The construction of multiple-tables, in the manner described above, is of help in obtaining factors. If we have formed our series of numbers, and have come on 21 in the series of multiples of 7, we get a reciprocal association of 21 and 7, leading to recognition of 7 as a factor of 21.

(7) The method provides an easy approach to the idea of decimals. For we may take 100 as our unit, and then each unit is divided into 100 components.

18. *Multiples of a unit.* From the idea of the series of natural numbers we pass to the idea of the corresponding multiples of a unit. Thus, if we take

4s. as our unit, we get the two parallel series 1, 2, 3 ... and 4s., 8s., 12s. ... These series would naturally be thought of as ascending; but it is more convenient to write them as descending:—

1	4s.
2	8s.
3	12s.
4	16s.
5	20s.
.	.

or horizontally:—

1	2	3 ...
4s.	8s.	12s. ...

I do not know which arrangement would prove to be the better, but I shall provisionally keep to the downwards one. The change from the upwards arrangement should not produce any difficulty.

The series can be amplified by the inclusion of fractional values.

19. *Arrangement of multiplication and division.* Suppose that from this series (adopting the downward arrangement) we take out the first pair and any other, say 7 ... 28s. Then we can insert these into a "form" made by crossing a vertical and a horizontal line, thus:

1	4s.		1	U
7	28s.	or	q	P

where U represents the unit, q the quotient (number of times the unit is taken), and P the product given by $P = qU$. The arrangement in the form is in fact merely a different way of expressing this latter relation. I find it very helpful.

There are here four entries: the "1" is permanent, and the omission of one of the others gives a particular kind of "sum." If the "28s." is omitted, we have a multiplication sum; if the "7," a "measuring" sum; if the "4s.," a "sharing" sum.

The entries could be arranged in other ways, e.g.:

1	7		1	q
4s.	28s.	or	U	P

I have retained the "s." in order to stress the fact that multiplication and division are essentially concerned with quantities; by omitting it we get a purely numerical statement.

20. *Proportion.* Now let us construct in the same way two series corresponding with one another and with the series 1, 2, 3 ... Suppose, for instance, that 1 lb. of tea costs 2s. Then we might take corresponding units in a great many ways, e.g. 1 lb. and 2s., or $\frac{1}{2}$ lb. and 1s., or 2 lb. and 4s., and so on. Thus we should have sets of series such as:

1	$\frac{1}{2}$ lb.	1s.	or	1	2 lb.	4s.
2	1 lb.	2s.		2	4 lb.	8s.
3	$1\frac{1}{2}$ lb.	3s.		3	6 lb.	12s.
.
.
.

If from a set of series of this kind we take out two pairs of corresponding quantities, *e.g.*

6 lb.	12s.
14 lb.	28s.

we have a proportion statement; and if we omit one of the four quantities, *e.g.* if we write down

6 lb.	
14 lb.	28s.

we have a proportion sum.

21. *Division by factors.* It is interesting to see how this method affects division by factors. This process is often useful for finding a quotient, either integral or approximate: the troublesome thing is the calculation of the remainder when the quotient is integral.

Suppose we want to divide 61459 by 168, using the factors 4, 6, 7, and let us regard it as a measuring division. My method depends on two principles.

(A)	
168	61459
.	.
.	.
1	quotient

(i) The framework of our calculation is as shown in (A). We work from the upper line to the lower by a series of divisions. The ratio of the numbers in the upper line is (apart from remainders) the same as that of the numbers in the lower line: the natural thing is to retain this ratio throughout the calculation. We therefore, as shown in columns (1) and (2) of (C), divide both 168 and 61459 by 4; both the resulting quotients by 6; and both the new quotients by 7.

(ii) The first division gives quotient 15364 and remainder 3. The ordinary method of dealing with this is to write both quotient and remainder in the second line. But this offends against our sense of neatness, since the 15364 are 4's and the 3 are 1's. We therefore write the 3 in the first line, connecting it with the 61459 by a dash, as in (B). If we read this dash as a minus sign, we have the statement that $61459 - 3 = 15364 \text{ times } 4$.

(B)	
168	61459—3
42	15364

(C)				
(1)	(2)	(3)	(4)	(5)
168	61459—3		1	3
42	15364—4		4	16
7	2560—5		24	120
1	365		Rem. = 139	

Acting on these two principles, we easily arrive at (C). The partial remainders 3, 4, and 5, resulting from divisions by 4, 6, and 7, are placed in column (3). The numbers 1, 4, 24 in column (4) are the units of these remainders, and can be obtained by division of 168 by the numbers in column (1). The total remainder consists of 3 1's, 4 4's, and 5 24's. The calculation of the total, 139, is shown in column (5); but this column can usually be omitted, the calculation being done mentally.

If we remove all remainders from column (2), we get the numbers 61320, 15330, 2555, 365. These are proportional to the numbers in column (1).

22. *Second Excursion. Initial principle.* I have said that we ought to have harmony between arithmetical relations, the way we think of them, and the symbols by which we represent our thoughts. A particular principle that follows from this is that our notation ought to lay the greater stress on the more important matters. Our second excursion deals with some deductions from this principle.

In stating a principle, I do not mean that it is a rule which must necessarily be acted upon. There may in any particular case be conflicting principles, the conflict leading to survival of one principle only, or to compromise.

23. *Symbol of multiplication.* One deduction from this principle has reference to the symbol for multiplication. A multiplication sum is usually written in this sort of form :

$$\begin{array}{r} \text{£} \quad \text{s.} \quad \text{d.} \\ 2 \quad 3 \quad 6 \\ \times \quad 4 \\ \hline \end{array}$$

But this only indicates the process which has to be performed : it does not represent what the pupil ought to be thinking about, namely 4 times £2 3s. 6d. And it is also a very imperfect indication of the process, since it only shows that the 6d. has to be multiplied by 4, nothing being said about the £2 and the 3s.

What we are thinking about may, with the present notation, be written either

$$4 \times \text{£2 3s. 6d.}$$

or

$$\text{£2 3s. 6d.} \times 4$$

according as we take "×" to mean "times" or "multiplied by". Of the two, the former seems to me much the better, since it directs attention to the most important part of the work.

I suggest, as I have suggested before, that we should reserve "×" to mean "times," and should denote "multiplied by" by a rather longer symbol. But, whatever we do about this, I think the symbol of multiplication ought to be directed to the most important part of the quantity that is to be multiplied.

I may add that, so far as I can make out, a common practice is to use the one sign "×" in the two senses, leaving it to the reader to find out from the context what the meaning of the sign is intended to be. This is a kind of variety that must be strongly condemned.

24. *Working from left (simple cases).* A wider deduction from the principle is that we should, as far as is possible and reasonable, begin with the more important parts of the calculation : in other words, that we should, as far as possible, work from left to right.

I am always surprised that there is so little done in this way in schools. I am not suggesting that all children should be taught to work from the left ; but I am sure there are some children who would like it and could do it. There are some simple numerical operations—multiplication by 2, addition of two numbers, and subtraction—which are frequently required and can, with a moderate amount of practice, be done by some persons more rapidly and with a greater sense of ease by this method than by the ordinary method. The actual calculation in this way has a charm which appears to be due partly to its rhythmic nature and partly to the fact that we are constantly looking ahead.

If to the three processes I have mentioned we add multiplication by 3 (which is more difficult), we have the material for easy multiplication by almost any number under 100.

As an example of a kind of calculation which is of frequent occurrence take the conversion of £37·4692 into pounds, shillings and pence.

$$\begin{array}{r} 374692 \\ 9384 \\ \hline 768 \end{array}$$

$$4608$$

Here, to multiply by 12, we first multiply by .2, by multiplying by 2 and shifting to the right: then add, and shift the decimal point. Some people, with practice, would find this quicker than multiplying by 12 in the usual way.

In teaching these methods to children to whom they are new, it will be found that a good deal can be left to the children themselves. Give them the general idea, and they will be able to work out the details. And don't worry overmuch about tidiness. The important thing is that the children should see for themselves that their first efforts are untidy and should want to remedy the untidiness.

25. *Multiplication and division generally.* From dealing with these simple cases we can go on to see where the principle leads us in reference to multiplication and division generally, whether of integers or of decimals. But this—beginning with the question of the placing of partial products in the multiplication of integers—is a very large question, and I cannot possibly enter upon it now. I may, however, say that I am glad to find that the so-called "standard form" method for multiplication and division of decimals is coming under criticism, and it is possible that the whole question may have thorough reconsideration. Here, at least, it is clear that there is room for variety and for further experiment.

The variety, however, is mainly of the vertical or progressive kind. The methods which are suited to an adult who has to do a good deal of calculation of this kind may not be suitable for the majority of children at school.

26. *Third Excursion. Rates.* Our final excursion is concerned with expressions such as

$$\frac{60 \text{ miles}}{1 \text{ hour}}, \quad \frac{28 \text{ apples}}{7 \text{ boys}},$$

and with statements such as

$$\frac{3 \text{ miles}}{4 \text{ minutes}} = \frac{66 \text{ ft.}}{1 \text{ sec.}}, \quad \frac{28 \text{ apples}}{7 \text{ boys}} = \frac{4 \text{ apples}}{1 \text{ boy}} = \frac{8 \text{ apples}}{2 \text{ boys}};$$

the first of these latter being a statement that motion at the rate of 3 miles in 4 minutes is at the rate of 66 ft. per sec., and the second being a statement that if 28 apples are distributed equally amongst 7 boys the distribution is at the rate of 4 apples per boy or of 8 apples per 2 boys.

Expressions of this kind are not much used in schools, but they are of very great importance in mechanics and physics, and they ought to find a place in the arithmetic course.

27. *Genesis of a rate.* To see how a rate arises, let us take a particular case.

Butter is sold at $10\frac{1}{2}$ d. the $\frac{1}{2}$ lb. What is the cost (a) of 3 lb., (b) of $4\frac{1}{2}$ lb., (c) of weight W ?

Here we take $\frac{1}{2}$ lb. as our initial unit. For (a) we divide 3 lb. by $\frac{1}{2}$ lb., getting 6 as the number of units; and we then multiply $10\frac{1}{2}$ d., the cost per unit, by this number. Thus the whole process is represented by

$$\frac{3 \text{ lb.}}{\frac{1}{2} \text{ lb.}} \times 10\frac{1}{2}\text{d.}$$

Similarly for (b) we get

$$\frac{4\frac{1}{2} \text{ lb.}}{\frac{1}{2} \text{ lb.}} \times 10\frac{1}{2} \text{ d.};$$

and for (c)

$$\frac{W}{\frac{1}{2} \text{ lb.}} \times 10\frac{1}{2} \text{ d.}$$

We see that $\frac{10\frac{1}{2} \text{ d.}}{\frac{1}{2} \text{ lb.}}$ is a constituent part of each expression, and we merely simplify our notation by writing this part as $\frac{10\frac{1}{2} \text{ d.}}{\frac{1}{2} \text{ lb.}} \times$. Then the answers to (a), (b), (c) are respectively

$$\frac{10\frac{1}{2} \text{ d.}}{\frac{1}{2} \text{ lb.}} \times 3 \text{ lb.}, \quad \frac{10\frac{1}{2} \text{ d.}}{\frac{1}{2} \text{ lb.}} \times 4\frac{1}{2} \text{ lb.}, \quad \frac{10\frac{1}{2} \text{ d.}}{\frac{1}{2} \text{ lb.}} \times W.$$

Thus the rate $\frac{10\frac{1}{2} \text{ d.}}{\frac{1}{2} \text{ lb.}}$ is an operator for converting the quantity 3 lb. or $4\frac{1}{2}$ lb. or W into its cost, just as a number or a fraction is an operator for converting a quantity into another quantity; and the method of operation by a rate $\frac{P}{V}$, where P and V are usually different kinds of quantity, is that, V and W being quantities of the same kind,

$$\frac{P}{V} \times W \equiv \frac{W}{V} \times P.$$

There is no difficulty in interpreting the operation in its second form, since $\frac{W}{V}$ is a number.

28. *Nature of a rate.* If the example just given is looked at in the way in which we look at proportion, we see that $\frac{1}{2}$ lb. butter and $10\frac{1}{2}$ d. are the initial units of a pair of parallel series showing respectively weight and cost. Thus a rate may be regarded as a symbol of the relation between two quantities which vary in simple proportion.

Another way of regarding a rate is as the result of a sharing division. If there are 28 apples to be distributed equally amongst 7 boys, we find the quantity given to each boy by dividing 28 apples by 7: to find the rate, i.e. the quantity per boy, we divide 28 apples by 7 boys, which gives 4 apples per boy.

If it be said that we cannot divide apples by boys, or that we cannot divide miles by minutes, the answer may be either (a) that we can properly have a symbol for anything which corresponds to a definite concept, or (b) that it is equally impossible to divide a quantity by a number.

There is really a strong similarity between the operations indicated by " $\frac{3}{4}$ of 60 mins." and by " $\frac{3 \text{ miles}}{4 \text{ minutes}} \times 60 \text{ mins.}$ " In the first, we, in the loose language of the text-book, "divide 60 mins. by 4," obtaining a quantity: and we multiply this quantity by the number 3. In the second we divide 60 mins. by 4 mins., obtaining a number: and we multiply 3 miles by this number.

29. *Reciprocal of a rate.* While $\frac{10\frac{1}{2} \text{ d.}}{\frac{1}{2} \text{ lb.}} \times 3 \text{ lb.}$ shows us the cost of 3 lb. butter, $\frac{\frac{1}{2} \text{ lb.}}{10\frac{1}{2} \text{ d.}} \times 3\text{s. 6d.}$ shows us the amount of butter that can be purchased for 3s. 6d., and the cost of this butter is $\frac{10\frac{1}{2} \text{ d.}}{\frac{1}{2} \text{ lb.}} \times \frac{\frac{1}{2} \text{ lb.}}{10\frac{1}{2} \text{ d.}} \times 3\text{s. 6d.}$, which must of course be 3s. 6d. Thus the reciprocal of a rate is found by inverting it, and the product—i.e. the joint effect—of a rate and its reciprocal is $1 \times$.

And it is obvious that a rate is not altered by multiplying both its elements—upper and lower—by the same number.

Thus we can play about with rates just as if they were fractions.

30. *What is expressed by a rate.* In many cases the symbol by which a rate is expressed may be taken as showing theness of the upper quantity in terms of the lower. The speed of a train measures its quickness. The symbol $\frac{10\frac{1}{2}\text{d.}}{\frac{1}{2}\text{lb.}}$ expresses, for the housewife, the costliness of butter in terms

of English money, while, for the dairyman, its reciprocal $\frac{\frac{1}{2}\text{lb.}}{10\frac{1}{2}\text{d.}}$ expresses the costliness of English money in terms of butter. The heaviness of a substance is expressed by dividing its weight by its volume; and so on.

31. *Successive operations.* We can have two or more operations in succession. Thus, in the above example, if we take 1 kilogram as = 2.2 lb., the amount of butter that can be bought for 3s. 6d. is

$$\frac{1\text{ kg.} \times \frac{1}{2}\text{ lb.}}{2.2\text{ lb.} \times 10\frac{1}{2}\text{d.}} \times 3\text{s. 6d.} = \frac{1\text{ kg.} \times \frac{1}{2}\text{ lb.} \times 3\text{s. 6d.}}{2.2\text{ lb.} \times 10\frac{1}{2}\text{d.}} = \frac{1 \times \frac{1}{2} \times 42}{2.2 \times 10\frac{1}{2}}\text{ kg.}$$

It should be observed that the operations are read from right to left. We begin with 3s. 6d., as being the quantity we are talking about. We multiply this by $\frac{1}{2}\text{ lb.}$ to convert it into lb., and multiply the result by $\frac{1\text{ kg.}}{2.2\text{ lb.}}$ to convert lb. into kilograms.

32. *Ultimate meaning of a rate.* There are two points which should by this time be clear.

(a) The first is that, in such an example as that just given, there is no difficulty in dividing kg. by lb., or lb. by d., because we are really working throughout with quantities of the same kind. Nominally we begin with 3s. 6d.; actually we begin with the weight of butter that can be bought with 3s. 6d.

(b) Thus, what we do, in each of a series of operations of this kind, is to multiply by something which is = 1.

(c) A point of practical importance may be added. If we make a mistake by introducing the reciprocal of a rate in place of the rate itself, we can always correct the mistake without working the whole thing again from the beginning.

If, in the above example, we had used $\frac{2.2\text{ lb.}}{1\text{ kg.}}$ instead of $\frac{1\text{ kg.}}{2.2\text{ lb.}}$, and had done some incidental calculations, all that it would be necessary to do, when we found that our dimensions were wrong, would be to multiply the result we had obtained by $\left(\frac{1\text{ kg.}}{2.2\text{ lb.}}\right)^2$.

33. *Importance of working with quantities.* This last sentence illustrates the importance of working arithmetic with quantities, not merely with numbers. A pupil who had written

$$\frac{2.2}{1} \times \frac{1}{10\frac{1}{2}} \times 42 \quad \text{or} \quad \frac{2.2 \times \frac{1}{2} \times 42}{1 \times 10\frac{1}{2}}$$

—the sort of mistake that is constantly made—would not see that his work was wrong, and therefore would be unable to correct it.

I referred, early in my address (§ 10), to the importance of keeping in touch with reality. I am glad that I have chanced, in concluding, to strike the same note.

I would add, by way of postscript, that I should be very glad to know of any experiments that are being made on the lines I have indicated, and that I should welcome any criticisms of my remarks.

W. F. SHEPPARD.

ARITHMETIC REVISED.

By W. MILLER, D.Sc.

If ab is the numerical value of the product of two factors a and b , and if h is added to a (for the purpose of bringing the product up to an assigned value P , or for any other reason), then bh must be added to ab . If, now, k be added to b (for the same or any purpose), the product must be increased by $k(a+h)$ where $(a+h)$ is the *new* value of the factor previously a . These facts are so elementary that they merit quoting only on account of the curious neglect of their systematic alternate use by the arithmetician.

Example 1. Multiply 2.38 by 5.76.

$$\begin{array}{r} 2 \times 5 = 10 \\ .3 \times 5 = 1.5 \\ 2.3 \times .7 = 1.61 \\ .08 \times 5.7 = .456 \\ 2.38 \times .06 = .1428 \end{array}$$

$$\text{Hence } 2.38 \times 5.76 = 13.7088$$

No advantages are claimed for the method in this example.

Example 2. Multiply 3.1415926... by 2.3457962... to (i) 4 decimal places or (ii) 5 significant figures.

We need not begin with one digit alone in each factor; we can make a beginning with *any* number of initial digits and by *any* method. A simple rule is to take the number of decimal places or significant figures *wanted* as the initial aggregate number in the two factors. We can begin with 3×2.3457 , or 3.1×2.345 , or 3.14×2.34 , or any other combination, and multiply by any method preferred. It will be found in practice that it is most suitable to take about the same number of digits in each factor.

We then find $3.14 \times 2.34 = 7.3476$ and immediately fix the decimal point.

It is proposed now to *complete* the work by the *above* method, introducing contractions:

$$\begin{array}{r} 3.14 \times 2.34 = 7.3476 \\ 3.14 \times .005 = 157 \\ .001 \times 2.345 = 23 \\ 3.141 \times .0007 = 22 \\ .0005 \times 2.3457 = 12 \\ 3.1415 \times .00009 = 3 \\ .00009 \times 2.34579 = 2 \end{array}$$

$$\text{Hence product} = 7.3695$$

Cross out end digit of each factor.

Now cross out one digit in each factor to the left of the previous ones.

Ditto again.

It will be observed that digits crossed out are multiplied only for the sake of the *carried* digit and that digits beyond them to the *right* are important *multipliers*.

In tabloid form the operation is as follows, but the finished operation obscures the process since digits are entered *one at a time* after the initial multiplication of 3.14 by 2.34:

As a contracted method this is the most economical of any, I hold, as regards thought or work or rules involved. The outstanding demerit of other methods is that the numbers are mutilated by *arbitrary rejection* of digits. In this method we carry on with digits beyond those crossed off until those digits *automatically* prove themselves *ineffective*.

To illustrate this point decimalise £1 17s. 7½d. and multiply by 79364.8266, getting the result to the nearest ¼d.

$$\begin{array}{r} 3.14159 \\ 2.34579 \\ \hline \text{Initial Stage} \left\{ \begin{array}{l} 6.28 \\ 942 \\ 1256 \\ 157 \\ 23 \\ 22 \\ 12 \\ 3 \\ 2 \end{array} \right. \\ \hline 7.3695 \end{array}$$

Examine also by this routine method the error in the product of 3.141×2.345 where these numbers have a possible error of $\pm .0005$. In such cases the possible error is given without trouble.

However, to return to the fundamental principle, if the product P is assigned, ab is a first approximation and $(P - ab)$ the "defect."

If, as in examples following, $k = h$, then

$$h(a + b + h) = \text{the defect};$$

h is then discovered by dividing the defect by $a + b$, i.e. by the sum of the factors found (or about to be found).

Example 3. Find two numbers differing by 2 whose product is 18.7362.

By trial the first approximations are 3 and 5. To maintain a difference 2 the digits following 3 and 5 must be the same (i.e. $h = k$), and these digits are found by dividing the defect by the sum of the factors just found (or about to be found).

3	$\times 5$	= 15	Defect	3.7362
.4	$\times 5$	= 2.0	"	1.7362
3.4	$\times .4$	= 1.36	"	.3762
.04	$\times 5.4$	= .216	"	.1602
3.44	$\times .04$	= .1376	"	.0226
.002	$\times 5.44$	= .0108	8	.0117
3.442	$\times .002$	= .0068	84	.0048
.0005	$\times 5.442$	= .0027	210	.0021
3.4425	$\times .0005$	= .0017	2125	.0004
.00004				

Hence the factors are 3.44254 and 5.44254.

The following is a tabloid form with the contracted methods shown in the multiplication above. The factors are written under each other, and the Italian method of subtraction is used. When the digits of the "dividend" have all been taken down, the two end digits (at that stage) are crossed out simultaneously and so on, as in the multiplication above, but in this case until the dividend is exhausted.

3.44254	18.7362
5.44254	3 7
	1 73
	376
	1602
	226
	117
	48
	21
	4

Since

$$3.44 \dots (2 + 3.442 \dots) = 18.7362$$

$$= -5.442 (2 - 5.44 \dots),$$

3.44254 and -5.44254 are the roots of the equation $x(2 + x) = 18.7362$.

The tabloid solution suggests the question as to whether there is any justification for reducing the solution of a quadratic equation to that of extraction of square root, particularly as the ordinary extraction of square root relapses into solutions of quadratic equations.

Unfortunately tabloid solutions tend to obscure the fundamental ideas. The first solution of *Example 3* shows that the solution is essentially one of building up a product—the product given by the sum of the second last column—and that the last column (the inverse process) is merely directive in the selection of digits for the factors. In the solution of higher equations, the last operation only is inverse, while there may be several direct processes preceding. We give next a more general example of the same type of quadratic equation with a view to studying the best method of building up such products as occur in the solution of equations of a general type.

Example 4. Solve $x(2.3146 + x) = 18.7362$.

Solution :

3	$\times 5.3146 = 15.9438$		Defect	2.7924
.3	$\times 5.3146 = 1.5943$	8	"	1.1980
3.3	$\times .3 = .99$		"	.2080
.02	$\times 5.6146 = .1122$	92	"	.0957
3.32	$\times .02 = .0664$		"	.0293
.003	$\times 5.6346 = .0169$	038	"	.0124
3.323	$\times .003 = .0099$	69	"	.0024
.0002	$\times 5.6376 = .0011$	2752	"	.0013
3.3232	$\times .0002 = .0006$	6464	"	.0007
.00008				

Hence the factors are 3.32328 and 5.63788.

In tabloid form we might proceed as follows, using Italian subtraction and contractions :

2.3146	18.7362 (3.3
3	2.7924
5.3146	1.1980
.3	.2080

and so on.

2.3146	18.7362 (3.3232
3	2.7924
5.3146	.2080
3.3	.0293
8.6146	.0024
.32	.0006
8.9346	
23	
8.9576	
32	
8.9906	

However, it is observed, as the fundamental principle suggests when $k=h$, that we might add the factors before multiplying by the new digit. Hence the following solution which should be compared step by step with the above solution :

Another example is set out at the end of the paper, without Italian subtraction.

This method might have been used for Example 3, as both equations are of the form $x(p+x)=q$ where p and q are positive.

It is unnecessary to deal with *square root* separately as this is a particular case of the above where $p=0$.

All quadratic equations with real roots can be put, by substitution of $-x$ for x if necessary, into the above form or $x(p-x)=q$.

Example 5. Solve $x(4.3872 - x) = 3.1749$.

This type can be changed into the previous type by the substitution of $2-y$ for x where 2 is the digit less than half of 4.3872. This substitution has the advantage of avoiding negative signs (or subtractions). However, the solution may be effected exactly as in Example 4 if due regard is paid to sign. In such a case, since $k=-h$, the best divisor is the difference of the factors just found (or about to be found). The difference in method is so slight that a worked example other than the diminutive one below is unnecessary.

The above provides systematic methods of solution of quadratic equations and of simultaneous equations of type $x \pm y = p$, $xy = q$. Evaluation of expressions such as $\sqrt{p^2 + q}$ and others, in which p and q are unwieldy numbers, is easily effected by putting $x + p = \sqrt{p^2 + q}$, so that $x(2p+x) = q$, solving for x and adding p . Even in finding factors for such expressions as

$$x^2 + 156x + 5184,$$

the fumbling methods of unsystematised trial and error can be replaced by direct certain system.

$$\begin{array}{r|l} 156 & 5184 \text{ (} 48 \text{ and } 156 - 48 = 108 \\ 4 & 464 \\ \hline 116 & 544 \\ 48 & 544 \\ \hline 68 & \end{array}$$

Contracted Division. Before proceeding to equations of higher degree we shall examine the method for ordinary division by contracted methods, one factor being fully given. An examination of the uncontracted method of multiplication will show that, at any stage, the number of digits in the factors is about half of those in the full product. Hence in dividing 5·378926 by 8·463927 we begin with half (or more) of the digits of the divisor, say 8·463. As in the case of multiplication, the first stage can be carried out by any method, giving a quotient ·635 and remainder ·004921. As we have now taken down all the digits of the dividend, we cross off two end digits of the factors and complete by contracted methods as below. No difficulty need ever be experienced in this method if, the decimal point being ignored, the digits of the divisor at any stage constitute a number at least ten times the quotient at that stage.

$$\begin{array}{r} 8 \cdot 463927 \\ \cdot 63512 \end{array} \left. \begin{array}{l} \cdot 004921 \\ 4349 \\ 117 \\ 104 \\ 19 \\ 15 \end{array} \right\}$$

According to the fundamental principle the defect is $bh + ak + hk$, and, to obtain an approximate trial divisor for h , bh should predominate over ak .

Example 6. Using the final tabloid form of Example 4, evaluate $x(x+3)-6$, given that the value of x is only gradually learned—digit by digit—in the course of the calculation, to be 1·74657.

This is actually a part of a solution of a cubic equation below. The calculation scarcely needs explanation if the previous pages have been read.

It should be noted that, at each stage, the latest increment of the function is given as well as the full value of the function at that stage. This is important in the case where this functional value has again to be multiplied by a growing number.

3	-6	1·74657
1	4	
4	-2	
1·7	3·99	
5·7	1·99	
·74	2576	
6·44	2·2476	
46	·03892	
6·486	2·28652	
65	325	
6·4925	2·28977	
	45	
6·5	2·29022	

Higher equations. It is proposed now to consider how the methods may best be extended to higher equations.

If
$$f(x) \equiv x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_{n-1}x + a_n,$$

we may write $f(x)$ thus :

$$\dots x [x \{x (x \cdot \overline{x + a_1 + a_2} + a_3) + a_4\} + a_5] + a_6 \dots,$$

working from within outwards so that the multiplier is always x . It is proposed to evaluate the function in this way for a value a of x which prospectively will make $f(x) = 0$.

As in the case of quadratic equations a trial divisor is necessary to determine the digits of a , and, as in that case, the trial divisor is $\frac{df}{dx}$ for the approximate value of a .

As a is evolved, successive values of

$$a + a_1, \quad a \cdot a + a_1 + a_2, \quad a(a \cdot a + a_1 + a_2) + a_3, \text{ etc.}$$

are calculated concurrently as part of the solution for a .

These approximations will themselves enable us to get closer approximations to a , and they will finally enable us to form immediately the equation of degree $n-1$ which gives the other roots, for

$$f(x) - f(a) = (x-a)[x^{n-1} + \overline{a+a_1}x^{n-2} + (a \cdot \overline{a+a_1+a_2})x^{n-3} + \dots],$$

as is easily verified on multiplication. So that, if ξ is any value of x for which $f(x)=0$, and a is one of these values, the other values are obtained from

$$\xi^{n-1} + \overline{a+a_1}\xi^{n-2} + (a \cdot \overline{a+a_1+a_2})\xi^{n-3} + \dots = 0 \dots A.$$

To carry on the calculation according to our fundamental principle, at every stage of the calculation of the coefficients of last equation, the increments should be given as well as the new aggregate value.

The final coefficient in equation A, used in our calculation, is dealt with in a slightly different way from the others as it is the means of giving new digits for a , but this is best shown in an example.

Example 7. Solve $x^3 + 3x^2 - 6x - 4 = 0$, giving the roots correct to five places of decimals.

As in other methods it is found tentatively that $a \doteq 1.7$. It is easily found that, for this value of x , $\frac{df}{dx} = 12.87$.

Example 6 gives a convenient form for the first stage of the calculation—but this form is not indispensable. The rest of the calculation is easily followed from the headings—the fundamental principle does the work.

Divisor 12.87.

	$a \cdot \overline{a+3} - 6$	a	$a(a \cdot \overline{a+3} - 6)$	Defect from 4
3	-6	1.74657		
1	4			
4	-2			
1.7	3.99			
5.7	1.99	$\times 1.7$	$= 3.383$.617
.74	.2576	$\times 1.7$	$= .43792$	
6.44	2.2476	$\times .04$	$= .08990$	
46	.03892	$\times 1.74$	$= 3.91082$.08918
6.486	2.28652	$\times .006$	$= .06772$	
65	.00325	$\times 1.746$	$= 3.91082$.08918
6.4925	2.28977	$\times .0005$	$= .06772$	
	.00045	$\times 1.7465$	$= 3.99226$.00774
6.49	2.29022	$\times .00007$	$= .00568$	
			$= 114$	
			$= 3.99908$.00092
			$= .00079$	
			$= 16$	
			4.00003	-.00003

Hence

$$a = 1.74657$$

$$a + 3 = 4.74657$$

$$a(a + 3) - 6 = 2.29022$$

The other roots are those of $x^3 + 4.74657x + 2.29022 = 0$.

If $x = -y$,

$$y(4.74657 - y) = 2.29022,$$

or $y = 2 - \xi$, i.e. $x = \xi - 2$,

$$\xi(4.74657 + \xi) = 3.20292.$$

The solution of the ξ -equation is :

.74657	3.20292 (1.45490
1	1.74657
1.74657	1.45635
1.4	1.25863
3.14657	19772
45	17983
3.59657	1789
54	1460
3.650	329
5	329
3.655	

Hence $\xi = 1.45490$ or -2.20147 ,
since the sum of the roots is -1.74657 .

Hence the roots of the cubic are

$$-1.54510$$

$$-4.20147$$

and

$$1.74657,$$

the sum being exactly -3 as it should be.

Extraction of cube root is only a particular case of the solution of cubic equations and need not be dealt with separately.

To find a root of a biquadratic equation little more work is required. In the second last column of the cubic solution the sum only of the two increments should be entered. An additional number has to be added in corresponding to the additional term. An additional column is then required for multiplication again by a and the work is completed as before. Again the cubic equation for the other roots can be written down at once and solved in the same way. If two of the roots are unreal, a quadratic equation is evolved, and from it the unreal roots can be found.

The same principle is applicable to equations involving other than algebraic functions as well as to higher algebraic equations.

1 Hume Street, Dublin, 29th Jan. 1929.

WILLIAM MILLER.

942. [M¹. 1.] An extension of the result given by Dr. Bromwich in *Math. Gazette*, xiv (1929), p. 285, may be of interest.

"Two lines drawn through any point O in fixed directions meet a given plane algebraic curve in the finite points $P_1, P_2, P_3, \dots; Q_1, Q_2, Q_3, \dots$. Then the ratio $(OP_1 \cdot OP_2 \cdot OP_3 \cdot \dots)/(OQ_1 \cdot OQ_2 \cdot OQ_3 \cdot \dots)$ is independent of the position of O , if and only if the tangents to the curve at the infinitely distant points in the given directions (if any) all coincide with the line at infinity."

HAROLD HILTON.

GLEANINGS FAR AND NEAR.

680. I know nothing whatever about the relation of 54 to Plato's number, and I strongly suspect that Rabelais knew no more than I do. It all has much the same effect on me as the *Hexeneinmaleins* in Faust. Goethe gave to his readers a nut which he found too hard for his own teeth to crack, and laughed in his sleeve at their attempts.—*Intermédiaire des Mathém.* ii. p. 103.

VOTING IN THEORY AND PRACTICE.*

By PROF. J. E. A. STEGGALL, M.A., F.R.S.E.

VOTING, in some form or other, must have been one of the activities of primitive man: and one can easily picture the cave-dweller expounding, perhaps with illustrations on the wall of his dwelling, some of the difficulties of theory and practice quite as earnestly as we shall consider the subject to-day, and, I am inclined to think, with no less success. The announcement of the results of any system was probably no less exciting than the clash of spear on shield that declared the verdict in Athens, or the public vote by a show of hands. Secret voting was only permitted when individual persons were directly concerned. It will be remembered that John Stuart Mill's judgment is in exact accord with the ancient practice: his illustration of the exceptional case being that of club membership, where the "blackball" of the old-fashioned ballot-box still represents the pebble of ancient Greece. In Rome voting "by classes" was frequently used, until corruption led to the domination of the wealthy; and a form of ballot by the tabella was used about B.C. 120.

In course of time various qualifications were introduced: in France residence, in Italy capacity to read and write, in Japan property—greater with illiteracy—the same in Portugal. In Great Britain parish relief was a disqualification for the recipient until now—but the details are endless.

There were many curiosities of the ballot—in the United States, not very long ago, there were 27 varieties of the cross on whose validity judges differed—in Australia there were 13 ways of placing the mark wrongly.

In modern times the counting of votes has always been a minor difficulty: machines have been introduced for automatic registration and enumeration, with the result that in one town (Buffalo, 1911) of 60,000 voters the results, with 100 candidates, were announced an hour and a quarter after the closing of the poll.

With this brief introduction we may consider first a few general principles, and next some special methods that are or may be used to fulfil them.

When one person has to be selected out of two it seems obvious that a single vote is decisive; and no doubt, as a final step, this is undeniable; but when electors are almost equally divided between two men, as to precedence, while at the same time the majority have a very high opinion of the other man, and the minority have a very moderate opinion of the first man, it is possible among reasonable men that a majority might, on reconsideration, waive their right—as commonly understood—in favour of a minority; and in many easily imagined cases it might be politic, or even wise, to do so.

The illustration is given here because a somewhat similar consideration will arise later in this discussion.

When there are more than two candidates for an office, not only is there much injustice in some of the methods in vogue, but there is a very considerable chance that those ordinary methods cannot give any result at all. Let us take a case of 15 electors and three candidates A, B, C; and suppose that the candidates are placed in the order A, B, C, by 4 electors and so on as below:

A, B, C	-	-	-	-	-	4 electors.
A, C, B	-	-	-	-	-	0 "
B, C, A	-	-	-	-	-	1 "
B, A, C	-	-	-	-	-	4 "
C, A, B	-	-	-	-	-	5 "
C, B, A	-	-	-	-	-	1 "

In a parliamentary election C would be elected. But in many elections the method is adopted of rejecting the man with the fewest votes, and taking a

* A lecture given at the Annual Meeting of the Mathematical Association, Jan. 8, 1920.

second vote between the remainder. In this case B will be elected by 9 to 6, assuming that there is none of that common trickery practised at many elections in which the second vote of some electors is the reverse of the first as between B and C. As a matter of fact neither of these results is just. A is the right man, for he is preferred to B by 9 to 6, and to C by 8 to 7.

In some elections, especially when there are three candidates, it is agreed that the so-far-successful candidate shall be put up against the man that was first eliminated; this, of course, tends towards a fair result. If this were done in the case just given, A, the proper man, would be elected. But what of this case now following? Let the votes be

A, B, C -	-	-	-	-	-	0 electors.
A, C, B -	-	-	-	-	-	4 "
B, C, A -	-	-	-	-	-	1 "
B, A, C -	-	-	-	-	-	4 "
C, A, B -	-	-	-	-	-	3 "
C, B, A -	-	-	-	-	-	3 "

on the first system considered. The first vote gives A, 4; B, 5; C, 6; and A is eliminated; between B and C the result is B 5, C 10; and C is elected; but if he is put up against A he is beaten by 7 to 8. In this case the votes of A against B are 7 to 8, of B against C 5 to 10, of C against A 7 to 8; and no finality is reached.

There are many philosophically minded writers, such, for example, as Carlyle and Ruskin, who in more or less definite terms point out that in their opinion it is hopeless to seek wisdom from aggregate folly; nevertheless it seems that we are tied down to the doctrine of the divine right of a majority; and consequently we must in any system provide for this right—if a majority exists. The application of this remark appears in the sequel.

Again, it is desirable that no opportunity shall be given for manœuvring by change of vote; and therefore if possible each elector's vote should be given only once—that is to say there should be no second election—subject of course to subsequent scrutinies of the one set of voting papers.

Again, when several positions of similar character have to be filled, a case of much greater difficulty than those hitherto considered where there was only one post involved, there should be some reasonable representation of minorities. Under some present and many past conditions a bare majority can fill up a large board entirely with its own representatives. Any sound system of voting should give to each group of voters a number of representatives proportionate to its own numerical strength—always assuming the accepted axiom that from aggregate folly wisdom may be born.

To meet these conditions the most important effort seems to have been that of Mr. Thomas Hare, who published in 1859 a small treatise entitled *Treatise on the Election of Representatives*. This treatise, which was followed by an appreciative and explanatory pamphlet by the late Professor Henry Fawcett, is the actual basis of the modern proportional representation method. Of course there have been other suggestions, among which the cumulative vote of James Garth Marshall had a long trial and an unlamented death. Of this and similar schemes Mill says that "though infinitely better than none at all, they are yet but makeshifts."

In the year 1882 the late Professor Nanson of Melbourne University drew up a careful examination of various systems of voting; and succeeded in introducing, at least into some of the Victorian elections, his own system.

It appears from subsequent pamphlets that the practical working presented some difficulties which were moreover increased, in the usual way, by the incapacity, ignorance and carelessness of voters, and the determined efforts of opponents whose interest it was to secure a reversion to what may be called the caucus principle.

The interest excited by Nanson's work is still alive, for only the other day Professor Sommerville issued two interesting and valuable papers of a somewhat abstruse kind dealing with the subject of preferential voting and based upon Nanson's original tract and the various later essays by G. Hogben, J. M. Baldwin, and himself. The two new papers, which are quite unsuited for quotation here, are to be found in the *Proceedings of the L.M.S.* ser. 2, vol. 28, pt. 5, and of the *R.S.E.* vol. 48, pt. 2, No. 12.

Before considering in detail the method called sometimes Borda's, sometimes Condorcet's, which is the basis of Nanson's, two elections may be touched upon. The first is that of the early thirteenth-century church. In the year 1215 Pope Innocent III. published the decretal "*Quia propter*," in which three methods are propounded. One of these "*Per Inspirationem*" was in effect the mere declaration of a decision of all the electors without difference of opinion. St. Ambrose was elected Bishop of Milan, while not yet baptized, by this method; Dr. Legg with good reason declares that this is an instance of election by popular tumult rather than according to the canon law; the same may be said of Hildebrand's election, as Gregory VII., to the Papal Chair.

A better instance is found in the election of John Islip (1500) as Abbot of Westminster. In this case Vaughan, the director of the chapter, asked the prior and convent how the election should proceed, to which they all answered, "By the way of the Holy Ghost." Whereupon William Lambard stood up and named John Islip, and immediately all the monks, John excepted, without any delay or discourse with one voice declared for Islip.

This method is practically that used to-day in the election of bishops when, after the reading of the *cong  d' lire*, the recommendation of a particular man by the sovereign is accepted with unanimous acclamation.

Although the Pope *may* still be elected "*per inspirationem*" the usual method is by scrutiny, "*per scrutinium*"; this was used in choosing William de Pickering as Dean of York in 1310.

The voting papers, which in the case of the papal election are described and illustrated fully by Wickham Legg in his *Ecclesiological Essays*, are very carefully though simply designed in order to preserve secrecy; and a majority of two-thirds on the single vote system is required; and until (each successive series of ballot-papers being burnt) this majority is obtained the election goes on again—as to how this majority comes to be attained ultimately Dr. Legg is I think silent.

The same method was, and perhaps still is, used for electing the President of the Royal College of Physicians, with this proviso; failing a two-thirds majority for any person the two names with the highest number of votes are then voted upon, and the winner is elected. It is unnecessary to discuss this method, obviously absurd in theory but possibly satisfactory in practice.

The last ecclesiastical method is by compromise; that is, by a small body of compromisers. In special cases one compromiser, outside the society, was entrusted with the nomination of its head. Matthew Parker was elected to the See of Canterbury by Nicholas Wotton the Dean; and in recent days Dr. Temple was elected Bishop of London in 1885 by Dean Church as sole compromiser nominated by the Dean and Chapter.

Quite recently there was an election to the Presidency of the Royal Academy, in which, according to the *Times*, the rule is for each academician to vote for one person and to send up to the king the names of those two that have the largest number of votes. Unless great care is taken in selecting the actual candidates on a short "leet," the result may be in complete opposition to the wishes and the judgment of the electorate.

The illustrations next presented are drawn from the actual practice of a large Educational Authority in Scotland. The number of members is 22, and the method of election in 1925 was, and I believe still is, that on which the first numerical illustration in this paper was based.

Suppose that there are three candidates, A, B, C; and that of the 22 members of the Authority nine place these candidates in the order B, A, C; six in the order C, A, B; five in the order A, C, B; and one in each of the orders A, B, C; C, B, A.

On the first vote for a single candidate A receives 6; B, 9; C, 7 votes; and A is at once and finally ruled out. The second vote then gives B 10, C 12; and C is elected.

Now as a matter of fact A is really preferred to B by 12 to 10, and to C by 15 to 7; he is therefore the proper person to be chosen, whereas he is ruled out at the first vote.

Moreover C is very doubtful as compared with B, for if each pair of candidates had been put up and voted upon A would have obtained 12 votes against B and 15 against C, or 27 in all; B would have obtained 10 votes against A and 10 against C, or 20 in all; C would have obtained 7 against A and 12 against B, or 19 in all: in fact C is the worst candidate, yet is elected if there are no changes in the votes; for I am assuming that each elector records at every stage the same opinion as to the relative order of candidates. To do otherwise is only to suggest reprisals and to make uncertainty greater. I therefore assume throughout that the voting is consistent at every stage.

Next suppose that there are four candidates A, B, C, D; I think it is clear that the more numerous the candidates the greater the chance of an unsatisfactory result. I proceed to an illustration which shows that of four candidates one may, under the present system, be elected to whom each of two others is preferred by a majority.

Suppose that the electors' opinions as to the order of the candidates is as shown below:

A, C, B, D	-	-	-	-	-	2 electors.
A, D, B, C	-	-	-	-	-	2 "
B, A, C, D	-	-	-	-	-	1 "
B, A, D, C	-	-	-	-	-	5 "
C, A, B, D	-	-	-	-	-	7 "
D, A, B, C	-	-	-	-	-	5 "

On the first vote A gets 4; B, 6; C, 7; D, 5 votes; A is therefore rejected. On the second vote B gets 6, C 9, D 7 votes; B is therefore rejected. On the third vote C receives 10 and D, who is elected, 12 votes.

Now an examination of the preferences given above shows that A is preferred to B by 16 to 6; to C by 15 to 7; to D by 17 to 5. He is in fact the best candidate in the opinion of the electors and has nevertheless been thrown out on the first vote. Again, B is preferred to C by 13 to 9, to D by 15 to 7. He is rejected on the second vote. Of course D is preferred to C by a majority; but again he is the poorest candidate for if each were put up against every other A would receive 48 votes in all; B, 34; C, 26; and D, 24 as shown in the scheme below:

	A	B	C	D	TOTAL.
A v. B	16	6	—	—	22
A v. C	15	—	7	—	22
A v. D	17	—	—	5	22
B v. C	—	13	9	—	22
B v. D	—	15	—	7	22
C v. D	—	—	10	12	22
	48	34	26	24	132

Thus the candidates are rejected exactly in the order of their merits as tested in the most exhaustive way and the worst is finally elected.

The case of five candidates A, B, C, D, E may give an even worse result. Let the preferences be

A, C, B, D, E	-	-	-	-	-	1
A, D, B, C, E	-	-	-	-	-	1
A, E, B, D, C	-	-	-	-	-	1
B, A, C, D, E	-	-	-	-	-	1
B, A, E, D, C	-	-	-	-	-	1
B, D, A, C, E	-	-	-	-	-	2
C, A, B, E, D	-	-	-	-	-	4
D, A, B, C, E	-	-	-	-	-	5
E, A, B, C, D	-	-	-	-	-	6

Then A is preferred to B, C, D, E by majorities of 14, 14, 8, 10; B is preferred to C, D, E by majorities of 12, 10, 10; C is preferred to D, E by majorities of 2, 6; yet A is rejected first, B next, C next, and D last. Thus E is elected. But on a vote between every pair of candidates A obtains 67; B, 53; C, 35; D, 33; E, 32 votes; or E is really the weakest candidate of all.

It is clear that when there are many candidates compared with the number of offices to be filled it will be desirable to reduce their number: in general there will be no difficulty in doing this, but a very curious case may arise; indeed did arise in 1872 at Carlisle. I am indebted to my old friend, Harvey Goodwin, long since dead, the Bishop of Carlisle and formerly a Senior Wrangler, for two hints on this subject. A list of 43 applications for a Diocesan Inspectorship of Schools was reduced to 5 without difficulty. The committee were then instructed to reduce the list of 5 to 3.

They did so by striking out the names of the two persons who in successive ballots were regarded as the "least eligible": that is to say, each member of the committee voted against one candidate only in each of two ballots. The result was that a man who had a majority against all the other candidates was struck out on the second ballot.

The bishop remarks that this candidate had a narrow majority against another, while his majority was qualified by the fact that the minority considered him to be the worst, while his majority considered the other man to be next to their favourite, a state of affairs that leads to the conclusion that in some cases a majority vote may be demonstrably a wrong vote and that the method of Borda without any qualifications is the fairest of all.

This method consists in putting up all the candidates in pairs, adding the votes for each, and placing the candidates in the order thus obtained. It seems impossible to add to the information used, for although there are differences of degree in preferences, it is as difficult to give due weight to them as it is to value the individual wisdom or judgment of the electorate; all that we can do, in practice, is to register the answer to the question, "Which of these two do you prefer?" Two details may be dealt with here. In the case of equality the single vote should be divided between the two persons concerned, each counting half a vote towards his total. Failure to make this arrangement will lead to injustice and will encourage manœuvring. The other detail is this—a candidate who has an absolute majority in his favour over every other candidate may be rejected. This, as Bishop Goodwin hints, is quite reasonable, though whether a majority will ever surrender its very doubtfully divine right is by no means clear. This matter is dealt with at the end of the paper. Meanwhile it is worth while to repeat formally the statement already made by implication that a candidate without being placed first by any elector may be preferred to every other candidate by a majority not always made up by the same individual voters.

In Borda's method it appears at first sight that the number of votings is excessive; but this is not the case. The result sought is obtained by every voter placing the candidates in order and indicating equalities. Any irregu-

larities—if they are not too gross—will be easily corrected by scrutineers, but this is a mere trifle. The essential point is that with 6 candidates, let us say, instead of 15 pairs being voted upon, the elector will place the numbers 1 to 6, or 0 to 5, or 6 to 1, or 5 to 0 opposite the names printed on his list or ballot paper. A moment's consideration shows that these last numbers give exactly the same result as would be derived from the 15 votings in pairs.

To so modify Borda's method as to give a majority its recognised privilege all that is necessary is that scrutineers should examine the voting papers and ascertain, as is quickly done, whether any candidate obtains a majority, not necessarily constituted in the same way, against every other candidate. If this is so he is elected; if not, the candidate with the highest number of votes will be elected.

To ascertain whether any candidate, say A, is preferred to every other candidate separately it is only necessary to examine whether he has a majority over B; if he has, try C; and so on. Suppose he is preferred to B and C and D but not to E, we then examine whether E is preferred to B, C, etc.: there is no need to examine B, C, D again because A is preferred to each of them, so E and F are the only candidates one of whom may possibly be preferred to each of the others. The process is very rapid. Of course it is more than likely that no candidate has a majority over each of the others; in this case the votes are added, and, subject to the usual procedure in cases of equality, the one with the largest aggregate is elected.

Two points still remain. Supposing that an elector cannot discriminate between two or more candidates, what is he to do? He must distribute their aggregate between them equally; this is equivalent exactly to giving them each half a vote when they appear in pairs. It is necessary that every voter should assign his full number of votes, for otherwise some injustice would ensue. I give two examples. Keeping to 6 candidates, suppose that an elector places A first, 3 second, C, D, E equal, F last, he would mark his paper thus:

A	-	-	-	-	-	-	5	
B	-	-	-	-	-	-	4	
C	-	-	-	-	-	-	3	
D	-	-	-	-	-	-	2	
E	-	-	-	-	-	-	1	
F	-	-	-	-	-	-	0	
								} equal

and the scrutineers would simply mark A, 5; B, 4; C, D, E each 2; and F, 0, making the total 15. Any voting paper in which the intention of the voter is clear would be corrected by the scrutineer.

Again, if an elector placed the six candidates thus, A first, then B, C equal, then D, then E, F equal, he would mark his paper thus:

A	-	-	-	-	-	-	5	
B	-	-	-	-	-	-	4	
C	-	-	-	-	-	-	3	
D	-	-	-	-	-	-	2	
E	-	-	-	-	-	-	1	
F	-	-	-	-	-	-	0	
								} equal,

and the scrutineers would give A 5; B, C each $3\frac{1}{2}$; D 2; E, F each $\frac{1}{2}$. The mean number is always the middle one of the bracketed group or the mean of the middle two. If it is desired to fill several posts with one vote (I have explained the process for one post only), the procedure is practically the same. The first post is filled by the candidate, if any, who is preferred by a majority to each of the others separately. If there is no such candidate, then the one who obtained the highest aggregate of votes is placed first. The list is again examined to ascertain whether any of the remaining candidates is preferred to each of the other remaining candidates; if so, he is elected to the second

post; and if not, the candidate who has the highest aggregate is elected, and so forth.

The other point is this: If an elector omits to mark some of the candidates, the scrutineers must assume that those omitted are last and equal in that elector's opinion and must adjust the votes accordingly. Thus, if an elector votes C, A, D, E equal, and ignores B and F, the scrutineer will give C, 5; A, D, E each 3; and B, F each $\frac{1}{2}$.

A further simplification can be made; if instead of marking the six candidates in order of merit from 5 to 0 they are marked 1 to 6, the result will be the same provided that the candidate with *lowest* total in place of *highest* total is preferred. This process is perhaps less confusing and is more in consonance with one's general ideas. The last two illustrations would then read thus:

A	-	-	-	1		A	-	-	-	1
B	-	-	-	2		B	-	-	-	2
C	-	-	-	3	and	C	-	-	-	3
D	-	-	-	4		D	-	-	-	4
E	-	-	-	5		E	-	-	-	5
F	-	-	-	6		F	-	-	-	6

Again, if voting papers are issued with the names of the candidates printed, a method precise and orderly but likely to prove impracticable, each elector must insert numbers opposite the names and the scrutineers may be put to much trouble in consequence of errors or irregularities. All that is really necessary is for each elector to write down the name, or initials, of the candidates in his order of preference indicating equalities by a bracket, and leaving to the scrutineers any necessary and usually simple adjustments. If, for example, an elector places the candidates in the order B; A, D equal; C, F equal; E; the scrutineers merely write 1 opposite B, $2\frac{1}{2}$ each opposite A and D, $4\frac{1}{2}$ each opposite C and F, and 6 opposite E; if an elector places only C; A, D, E equal; and ignores B and F; the scrutineers will mark C, 1; A, D, E each 3; B and F each $5\frac{1}{2}$.

Finally, to show on a small scale the whole process I give some illustrations when there are 4 candidates A, B, C, D, and seven electors P, Q, R, S, T, U, V.

I. Suppose the electors vote thus:

P	Q	R	S	T	U	V
A	A	A	A	B	B	C
B	C)	B)	B	C	D	A
C	B)	D)	C)	D	C	B
D	D	C)	D)	A	A	D

No assignment of votes is needed as A is placed first by a majority and is therefore elected.

II. Suppose they vote thus:

P	Q	R	S	T	U	V
A	A	A	B	B	C	D
B	C	C	C	C)	D	C
C	B)	D	A	D)	A	B
D	D)	B	D	A	B	A

Here A is preferred to B by 4 to 3; but he is preferred to C by 3 to 4; hence he falls out on the first scrutiny. B also falls out as he is already beaten by A. C is preferred to B by 4 to 3; and to D by 6 to 1. As he is also preferred to A by 4 to 3 he is the successful candidate.

III. Suppose they vote thus :

P	Q	R	S	T	U	V
A	A	A	B	B	C	D
B	C	C	C	C	D	B
C	B	B	A	D	A	C
D	D	D	D	A	B	A

Here A is preferred to B by 4 to 3, but to C by 3 to 4. Hence neither A nor B is preferred to each of the others. C is preferred to B by 3 to 4, so C is ruled out ; and D is preferred to A by 3 to 4, so that no candidate is preferred to each of the others by a majority. The scrutineers therefore insert the votes thus :

	P	Q	R	S	T	U	V	TOTAL.
A	1	1	1	3	4	3	4	17
B	2	3½	3	1	1	4	2	16½
C	3	2	2	2	2½	1	3	15½
D	4	3½	4	4	2½	2	1	21

Thus C is elected, for the meaning of the result is that in voting between every pair of candidates he would get the *largest* number of votes.

J. E. A. STEGGALL.

943. [K¹. 6.] *The treatment of the conic by point and by line coordinates.*

The conic is a curve of degree two and class two. Hence its treatment by point coordinates, and its treatment by dual or line coordinates run parallel, neither offering any substantial advantages over the other in the treatment of projective properties. While this fact is well known and pointed out in text-books on the Geometry of Conic Sections, the large amount of attention which the point conic secures at the expense of the line conic in these books would lead one to suppose that point coordinates are specially suited for the discussion of metrical properties. It seems to me that the exact opposite of this is true, and that metrical properties may be expected to take a simpler shape when the conic is given by its tangential rather than by its point equation ; for the Absolute which determines the character of the metrical field develops a distinctly tangential singularity in Euclidean space. Thus, in the plane the Absolute appears as a pair of points, which is the simplest type of a degenerate line conic, while pointwise, it is doubly degenerate, and exists as a repeated line. Further, we may pass from the point pair to the repeated line, but not *vice versa*.

As an illustration consider the equation giving the lengths of the principal axes of the conic :

$$a^2 + 2hlm + bm^2 + 2gl + 2fm + c = 0.$$

$$(ab - h^2)\Delta + \Delta^2 = 0.$$

The equation is of order three in the coefficients, and is in fact :

$$r^4 c^2 + r^2 c (bc + ca - f^2 - g^2) + \Delta = 0,$$

where Δ is the discriminant $abc + 2fgh - af^2 - bg^2 - ch^2$.

On the other hand, for the conic :

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

the axes are given by

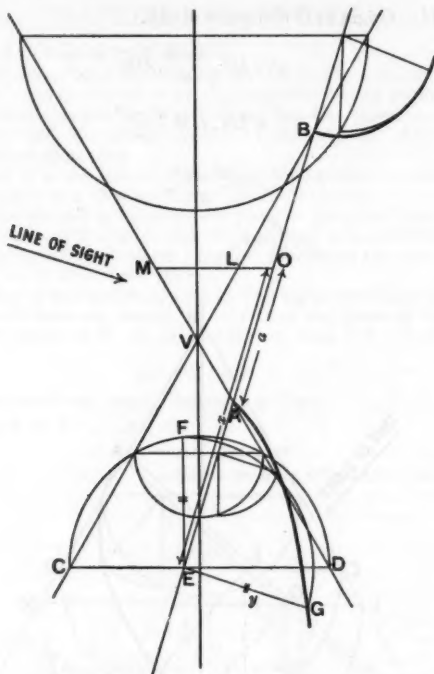
$$r^4 (ab - h^2)^3 + r^2 (a + b) (ab - h^2) \Delta + \Delta^2 = 0,$$

which involves the coefficients in the equation of the conic in the sixth degree.

If this is typical of most metrical properties, there seems to be a strong case for the tangential equation of the conic receiving a good deal more attention in text-books than it does at present.

The Presidency College, Madras.

A. NARASINGA RAO.



b , in both cases, being g.m. of LO , OM .

A. HINCKLEY.

945. [O. 2. c.] *Curvature of a Roulette. Elementary Proof of a Formula.*

An equilateral polygon G_1 inscribed in a curve C_1 rolls on another fixed equilateral polygon G_0 inscribed in a curve C_0 , so that the sides of G_1 coincide successively with the sides of G_0 , the rotations being about the successive angular points of G_0 .

A point P , fixed relatively to G_1 and C_1 , moves from Q to R when a rotation 2α takes place about A , and moves from R to S when a rotation 2β takes place about B , AB being one side of G_0 .

The "axes" of the pairs of points Q, R and R, S are AO and BO respectively.

Then $\hat{A}RB = \hat{A}OB + \alpha + \beta$, $\hat{O}AR = \alpha$, $\hat{R}BO = \beta$.

We take as the standard case that in which the concavity of G_0 and also that of G_1 are towards P .

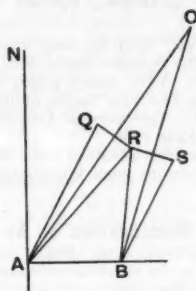
$$\text{Hence} \quad \sin \hat{A}RB - \sin \hat{A}OB = 2 \sin \frac{\alpha + \beta}{2} \cos \frac{\hat{A}RB + \hat{A}OB}{2}, \dots\dots\dots(1)$$

$$\text{But} \quad \sin \hat{A}RB = \frac{AB}{AR} \sin \hat{A}BR, \text{ and } \sin \hat{A}OB = \frac{AB}{AO} \sin \hat{A}BO.$$

Hence, by (1),

$$\frac{\sin \hat{A}BR}{AR} - \frac{\sin \hat{A}BO}{AO} = \frac{2}{AB} \sin \frac{\alpha + \beta}{2} \cos \frac{\hat{A}RB + \hat{A}OB}{2}, \dots\dots\dots(2)$$

We now suppose the common value of the lengths of the sides of G_1 and G_0 to diminish towards zero as limit. The limits of G_1 and G_0 are then the



curves C_1 and C_2 . The limit of the polygon ... QRS ... is the roulette described by P when C_1 rolls on C_0 , and the limit of O is the centre of curvature of this roulette corresponding to A as point of contact of C_1 and C_0 .

If $2a_1$ is the change of direction in G_1 at the point A and $2a_0$ that in G_0 , then $2a = 2a_1 - 2a_0$, if as in the figure $a_1 > a_0$. Now on taking the limit for $AB \rightarrow 0$, $\sin \hat{ABR} \rightarrow \cos \phi$ and $\sin \hat{ABO} \rightarrow \cos \phi$, where ϕ is the acute angle AP makes with the common normal of C_1 and C_0 at A . Also $AR \rightarrow AP \equiv r$ and $AO \rightarrow r + \rho$ where ρ is the radius of curvature of the roulette at P .

Further, $\frac{\beta}{\alpha} \rightarrow 1$ unless C_1 or C_0 has a discontinuity of curvature at A .

$$\text{Thus } \frac{2 \sin \frac{\alpha + \beta}{2}}{AB} = \frac{\sin \frac{\alpha + \beta}{2}}{\frac{\alpha + \beta}{2}} \cdot \frac{\alpha + \beta}{2a} \cdot \frac{2a_1 - 2a_0}{AB} \rightarrow 1 \times 1 \times \left(\frac{1}{\rho_1} - \frac{1}{\rho_0} \right),$$

where $1/\rho_1$ is the curvature of C_1 and $1/\rho_0$ the curvature of C_0 at A .

Hence in the limit (2) reduces to

$$\cos \phi \left(\frac{1}{r} - \frac{1}{r + \rho} \right) = \frac{1}{\rho_1} - \frac{1}{\rho_0}, \dots \dots \dots (3)$$

which is a well-known formula.

This formula applies to all the cases that can occur, besides concavity rolling on concavity (as in our figure), i.e. convex on concave, convex on convex and concave on convex, provided we keep to the rule that ρ_1 and ρ_0 are positive or negative according as the concavities of C_1 and C_0 at A are towards, or away from, P , and that ρ is positive or negative according as the concavity of the roulette at P is away from or towards A .

Corollary. If C_1 is fixed, and C_0 rolls on it, and $1/\rho'$ is the curvature of the roulette which P then describes, we have to substitute ρ' for ρ and interchange ρ_1 and ρ_2 in formula (3). This gives

$$\cos \phi \left(\frac{1}{r} - \frac{1}{r + \rho'} \right) = \frac{1}{\rho_0} - \frac{1}{\rho_1}, \dots \dots \dots (4)$$

From (3) and (4) we deduce

$$\frac{1}{r} - \frac{1}{r + \rho'} + \frac{1}{r} - \frac{1}{r + \rho} = 0,$$

which implies that the centres of curvature of the two roulettes at P are harmonic conjugates with respect to the points A , P .

R. F. M.

REVIEWS.

The Slide Rule. By G. A. GUNN. Pp. 40. n.p. 1928. (E. & F. N. Spon, Ltd., Haymarket, S.W.)

This little book aims to give "Step by step instruction" in the use of the Slide Rule, more especially the System Reitz Slide Rule. Much information is given in the limited space. Very many pages are devoted to the section on the Solution of Triangles, but two pages including four examples all on Interest "must suffice" for "Commercial Calculations (which) are greatly facilitated by the use of the slide rule."

From the beginning the writer assumes that the reader has an intimate knowledge of logarithms; without such knowledge much in the book would be very difficult to follow. E. J. ATKINSON.

Trisections of Angles; Rectification of Arches; Multiplication of Cubes. By FRANCESCO PAOLINO. n.p. 1928. (New York.)

A set of diagrams intended to be taken as substantial additions to geometry, since they are copyrighted. The first construction for trisection of angles is that used by Kempe in his *Linkages* (now out of print, I believe) to construct a trisecting compass; the second construction is on the same lines as that published by Mr. Maskelyne in the *Phil. Mag.* some years ago, and promptly shown to be inaccurate by several correspondents. If these constructions are not meant to be anything else but approximate trisections, why not use a protractor?

The duplication of the cube: of the fourteen diagrams which show "some of the many simple methods by which it is possible to proceed to the construction," those that can be followed without accompanying letterpress depend upon a well-known figure and its properties. $ABCD$ is a square, M is the middle point of DC ; CB, AB are produced to F, E , respectively, and P is the middle point of BE , where F, E are so chosen that $PF=PA$, and MP is parallel to AF . If this figure is drawn, $BF^3=2AD^3$, and $AE^3=4AD^3$. But there is no geometrical construction for the points E, F .

The construction for the multiplication of the cube is theoretically correct, and is ingenious, but is merely an extension of one of the diagrams noted above, a method of constructing this figure being given; but the construction is only tentative, the essential part of it being: Given two straight lines OX, OY , at right angles, X being a fixed point, and also a circle touching OX , it is required to draw a tangent to the circle, cutting OX in M and OY in N , so that $MN=MX$. J. M. C.

Mathematical Tables and Formulas. By P. F. SMITH and W. R. LONGLEY. Pp. vi+66. 8s. 1929. (Wiley & Sons; Chapman & Hall.)

The tables in this book are of squares and cubes, square roots and cube roots, reciprocals, sines, tangents, secants, and circular measure, the six circular functions and degree measure of angles in radians, logarithms and antilogarithms, logarithmic sines and tangents, exponential and hyperbolic functions, and Napierian logarithms. If the tables, most of which are to four significant figures, are accurate, I can only say that the collection is exactly what we have been wanting.

For the formulas, which occupy the last twenty-five pages, I feel no enthusiasm. Except in the table of integrals, they are almost all elementary results, or definitions, which no student can conceivably need to look up: $n! = 1 \cdot 2 \cdot 3 \cdot 4 \dots (n-1)n$; $\sin(\pi - A) = \sin A$; transformations from polar to rectangular coordinates; Taylor's series (with no remainder term);

$$\frac{dc}{dx} = 0, \quad \frac{dx}{dx} = 1, \quad \frac{d}{dx}(cv) = c \frac{dv}{dx}.$$

It is not to be disputed that a table of integrals is often useful: of the 180 formulas in the table here, there are very few which any of us carry in our heads; whether there is room for another small table is a more open question. The explicit addition of an arbitrary constant in every formula is somewhat childish, and it is disappointing to find again in the elementary

integrals the classical mistakes that have been so often pointed out. There are no results which involve a pair of quadratic functions, and no reduction formulas which involve one general quadratic function, but there are many reduction formulas which involve a quadratic function that lacks either the middle term or the constant term. No definite integrals are given.

The printing of the numerical tables and of the table of integrals is clear, but that of the formulas is unskilful. The binding is good, and there is a thumb-nail index. Were it not for the high price, the book could be recommended warmly for the sake of the tables, with the warning that without painful mutilation it could not be admitted into an examination room when a knowledge of the processes of integration was to be tested. E. H. N.

(1) **Lesehefte zur Mathematik.** Heft I.: Aus der Griechischen Mathematik. Ausgabe A. Griechische Texte. Pp. 32. Ausgabe B. Pp. 34. Edited by A. CwZALINA. 0.80 RM. each. (Hirt, Königsplatz 1. Breslau.)

(2) **Archimedes.** By F. KLIEM and G. WOLFF. Pp. viii + 142. M. 3. (Salle, Berlin.)

(3) **Fixstern Beobachtungen.** By J. PLASSMANN. Pp. viii + 120. M. 3.40. (Salle, Berlin.)

(4) **Rechnen und Algebra.** By H. WIELEITNER. Pp. viii + 75. M. 2. (Salle, Berlin.)

(5) **Wetterkarte und Wettervorhersage.** By B. TZSCHIRNER. Pp. viii + 62. M. 1.80. (Salle, Berlin.)

The classics will never be thrown into the sea, even, as a wag suggested—*ad usum Delphini*. "I should be very sorry," wrote Sir W. Rowan Hamilton to Salmon, "to lose the pleasure and the profit of reading occasionally in their own noble language the writings of the Greek geometers. . . . It has amused me to fancy sometimes a demonstration of Apollonius printed in one column of a book; a Cartesian investigation of the same theorem in another column, . . . and a quaternion calculation in the fourth." These are giants at play. But one need not be a giant to reap pleasure and profit from following their example. "I have long held that, if the study of Greek and Euclid be combined by reading at least part of Euclid in the original, the two elements will help each other enormously." Thus wrote Sir Thomas Heath nine years ago in the Preface to his *Euclid in Greek. Book I*. Such series as Ostwald's *Klassiker* and the *Alembic Club Reprints* are evidence of the importance attached to an acquaintance with the original papers which are landmarks in the development of a subject. The *Quellenbücher* have made an entrance into German schools in the form of German translations from the original. We have heard, so far, no such howl as Sir George Greenhill anticipated* "at the suggestion that some historical interest should be inculcated, say in the works of Hero, Archimedes, Archytas, and in the original Greek. . . . When we look round our great manufacturing establishments we see the managing director a product of Oxford Greats."

(1) Dr. Cwzalina introduces his selections from Euclid, Apollonius, Diofantus and Pappus by short historical sketches of the men and their work (pp. 1-11). We thus have the Greek text (the paper shows up well the pleasant type that is used) of Euc. I. 4; I. 22; I. 47; II. 14; IV. 10; VII. 1, 2 (pp. 12-17); of Apollonius I. 11; I. 52; II. 52 (pp. 18-22); Archimedes, an instance of the method by exhaustion and the famous Cattle Problem (pp. 22-24); Diofantus, No. 1, p. 131; No. 8, p. 133; No. 8, p. 144; No. 19, p. 151 (pp. 24-26); Pappus, the generalisation of I. 47, and a specimen of his treatment of an isoperimetric problem (pp. 26-28); finally from Cleomedes a passage in which the astronomer challenges Eratosthenes' calculation of the meridians of Syene (Assouan) and its distance from Alexandria (pp. 29-30). A short but useful vocabulary of Greek terms closes this catholic selection. Ausgabe B consists of a German translation of the selected passages. The low price of the monograph is to be noted.

**Math. Gazette*, x. No. 154, May 1921, p. 332.

(2) Drs. Kliem and Wolff have written the first volume of a new series for schools, etc., entitled "Mathematisch-Naturwissenschaftlich-Technische Bücherei," and have taken Archimedes as their subject. The first 14 pp. deal with mathematics before the time of Archimedes. Then after a short biographical sketch comes a series of chapters dealing with his work on Statics, Hydrostatics, Numbers and Algebra, Polyhedra, the Cone and Cylinder, Conics and Spirals, closing with an account of Heiberg's discovery of 1906. There are 64 figures and illustrations, and the whole series is well printed and on good paper. Dr. Kliem has translated into German Sir Thomas Heath's *Archimedes*.

(3) The second volume, by Prof. J. Plassman, deals with the "Fixed Stars," and is a model of what such an introduction should be. The eleven plates, in a cover at the back of the book, are beautifully reproduced.

(4) To Dr. Wieleitner's little volume we turn with particular interest. He provides his readers with German translations from early investigators, and has managed to crowd into his pages quite a remarkable number of interesting passages. Jehan Adam (1475), Johann Böschenteyn (1514), al-Khowārizmi, Adam Reise (1524), Rudolf (1525), Grammateus (1518), are the earlier of the twenty-three authors from whom quotations are given.

(5) Band V. is by B. Tzschiner and deals with Weather Forecasting. The twenty illustrations to this volume are simple and really instructive.

Altogether the series are most attractive, and the volumes numbered 1, 2, and 4 above are certainly worth the attention of mathematical teachers in this country.

681. . . . How do the dear girls go on ? I would have them taught geometry, which is of all sciences in the world the most entertaining : it expands the mind more to the knowledge of all things in nature, and better teaching to distinguish between truths and such things as have the appearance of being truths, yet are not, than any other. . . .

How would it enlarge their minds, if they should acquire a sufficient knowledge of mathematics and astronomy to give them an idea of the beauty and wonders of the creation !—From a letter of Admiral Lord Collingwood to his wife, 16th June, 1806.

682. I have seen this book [*Probability*. Useful Knowledge Library] sold with my name upon the cover as author. To this I have no objection, except my knowledge of the fact that it is the joint production of Sir John Lubbock and Mr. Drinkwater-Bethune.—De Morgan, *Arithmetical Books*, 1847, footnote, p. 106.

683. Loving De Morgan, and feasting on his books, . . . I naturally asked Sylvester for his judgment. He answered me in an epigram :

De Morgan did not write mathematics ;

He wrote *about* mathematics.

—G. B. Halsted (*The Monist*, circ. 1890).

684. "Mansel was great, . . . Tommy Short was brilliant : greatest and most brilliant was Henry Smith of Balliol. Did you ever hear his comment on the mathematical papers of two of his friends ? . . . 'How did you get on to-day, Mr. Brown ?' he asked.

"Brown produced the paper of Questions : 'In the Euclid I did that, and that, and that : I left out those, and I had a shot at those. In the Algebra I did those, and left out those.'

"'Oh yes,' said Smith, without faintest comment : 'and Mr. Jones, what did he do ?'

"'In the Euclid he did those, and left out those ; in the Algebra, he did those and left out those.'

"'Oh yes,' said Smith with increased politeness : 'then I should think he would be ploughed too.'"—E. M. Sneyd-Lynnersley, *Some Passages in the Life of one of the H.M.I.* p. 96. [Per Mr. P. J. Harris.]

MATHEMATICAL ASSOCIATION.

YORKSHIRE BRANCH.

THE Spring Meeting of the Yorkshire Branch of the Mathematical Association was held in the Mathematical Department of the University of Leeds on Saturday, 9th February, 1929. Two new members were proposed and 40 people were present. Mr. W. Peaker, B.Sc., was in the chair. Professor J. Proudman, F.R.S., gave a most illuminating paper on "The effects of capes, bays and islands on local tides." A discussion on the motion, "This Branch of the Mathematical Association is convinced that a pass in the Mathematics and Science group is an essential part of a School Certificate," was opened by Miss I. M. Mathews and Miss M. A. Hooke, M.A., Grammar School, Bradford.

The Summer Meeting was held on 11th May, 1929, at Queen Ethelburga's School, Harrogate. There were 42 members and friends present. Two new members were proposed. The Head Mistress (Miss E. L. Young, M.A.) extended a hearty welcome to all the members of the Yorkshire Branch. A most interesting paper was read by Professor Paul Barbier, M.A., University of Leeds,—"Notes on the History of Mathematical Terms in the vulgar tongues of Western Europe (French and English)." J. H. Blacklock, Esq., M.A., Senior Mathematical Master, The Grammar School, Rotherham, read a fascinating paper on "The Planetarium at Berlin." The speakers were heartily thanked by the Chairman (W. Peaker, Esq., B.Sc.) and Professor W. P. Milne, M.A., D.Sc., and Professor S. Brodetsky, M.A., Ph.D.

On Saturday, 16th March, 1929, the Yorkshire Branch Dinner was held at the Queen's Hotel, Leeds. There were 49 members and friends present. The Astronomer Royal (Sir Frank W. Dyson, M.A., LL.D., D.Sc., F.R.S.) was the guest of the Branch. Mr. W. Peaker, B.Sc., was in the chair. Professor S. Brodetsky, M.A., Ph.D., proposed the Toast, "Our Guest," which was replied to by The Astronomer Royal. Mr. W. F. Beard, M.A., M.Sc., proposed the toast, "Women Mathematicians," which was ably replied to by Miss K. Reeve. The toast to the Chairman was proposed by Professor W. P. Milne, M.A., D.Sc., and Mr. Peaker responded.

ARTHUR B. OLDFIELD, Hon. Sec.

MANCHESTER AND DISTRICT BRANCH.

REPORT FOR 1928-1929.

THE number of members of the Branch is now 98, of whom 22 are members of the parent Association. In general, four meetings are held during the year, of which one is arranged jointly with the University Mathematical Society. The other three take place at Manchester High School for Girls, by kind permission of Miss Clarke.

During this session the following papers have been read :

Oct. 30th, 1928. "That we can be too thorough."—T. DENNIS.

Dec. 4th, 1928. "Gravitation." (Lantern lecture).—PROFESSOR BRODETSKY.

Jan. 29th, 1929. Debate: "That the study of Geometry is valuable rather for logical training than for instruction in the properties of space." For: J. H. DOUGHTY. Against: E. E. DAVIES.

Feb. 20th, 1929. (At the University.) "S. Ramanujan, the Indian mathematician, 1887-1920."—PROFESSOR G. N. WATSON.

At the first meeting the officers for the year were elected, and there was a detailed discussion of the papers set in mathematics at the 1928 examinations of the Northern Universities Joint Matriculation Board.

M. O. STEPHENS (Hon. Sec.).

SIXTH ANNUAL REPORT OF THE QUEENSLAND BRANCH OF THE MATHEMATICAL ASSOCIATION.

THE Annual Meeting was held at the University on Friday, 25th March, 1927. The Annual Report and the Balance Sheet were presented to the meeting and were adopted, after which the election of officers for the ensuing year took place. As this meeting was very close to the two-hundredth anniversary of the death of Newton, Professor Priestley's presidential address was on the subject "Some Aspects of Newton's Work."

During the year three General Meetings were held: at the first of these, on Friday, 20th May, Mr. S. Stephenson, M.A., read a paper on "The Lighter Side of Mathematics." The second meeting was held on Friday, 5th August, at which Miss E. Raybould, B.A., read a paper entitled, "A Glance at the Foundations of Mathematics." Through various unavoidable causes the attendance at this meeting was small, and Miss Raybould, on request, agreed to read the paper again at a future meeting. At the third meeting, held on 28th October, Dr. E. F. Simonds, M.A., gave an address on "Groups."

The number of financial members of the Society is twenty-five, of whom thirteen are full members of the Mathematical Association: the membership thus is practically the same as for last year. Copies of the *Mathematical Gazette* have come to hand regularly and are circulated among Associate members. The "Newton" number issued by the Mathematical Association has been received and is also being circulated. The Statement of Receipts and Expenditure shows a credit balance of £7 16s. 8d., which is an increase of £1 6s. 8d. on that of last year.

30th March, 1928,

J. P. MCCARTHY, Hon. Sec. and Treasurer.

THE LIBRARY.

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THE CATALOGUE.

A *Second List of Books and Pamphlets in the Library* was issued to members with the last number of the *Gazette*; this and the *First List* form a Catalogue complete to the end of 1925. A third list, to extend to the end of the present year, will be prepared as soon as possible. Thanks to the collection of pamphlets left by Professor Genese, this next list will be of about the same size as its predecessors, but since it should be less laborious to check, the Librarian hopes that there will be less delay in its production. Meanwhile the items which it will contain, other than school-books and pamphlets, are recorded in acknowledgments in the last volume and the current volume of the *Gazette*.

NOTICE.—*Prof. Neville is visiting South Africa this summer, and from July until the beginning of October correspondence relating to the Library should be addressed to Mr. F. BEAMES, 7 Mansfield Road, Reading, who will have access to the shelves.*



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